Limit preservation properties of the greatest semilattice image functor

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Abstract
We study what kinds of limits are preserved by the greatest semilattice image functor from the category of all semigroups to its subcategory of all semilattices.

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1 Introduction

The most important tool in the description of the structure of semigroups is, arguably, the greatest semilattice decomposition. Many important kinds of semigroups can be characterised by the behaviour of the classes of this decomposition. What makes the greatest semilattice decomposition so special is the fact (detected by Tamura [8] in 1956) that every class of this decomposition is a semilattice indecomposable semigroup (that is, it does not map homomorphically onto any semilattice with more than one element). As was shown also by Tamura [9] in 1966, no other non-trivial variety of semigroups enjoys a similar property. (Let us note also that, in view of their indecomposability, the classes of the greatest semilattice decomposition of a semigroup appear like the connected components of a topological space; indeed, the two cases can be presented as special instances of the same phenomenon, see [6].)

Assigning to every semigroup its greatest semilattice image induces a functor $D$ (a reflection) from the category of semigroups to the category of semilattices, which is left adjoint to the inclusion functor of semilattices to semigroups. In view of the importance of the greatest semilattice decomposition, it is surprising that the properties of this reflection functor have not been investigated so far. This is what will be done in the present paper.

Being a left adjoint, the functor $D$ preserves all colimits, hence the natural question which poses is what kinds of familiar limits are preserved by $D$. The kinds of limits we are going to consider are products, finite products, monomorphisms, pullbacks and equalizers. At the end of the paper we put a table which summarizes the situation with them.

Among the limits to be investigated, pullbacks play a special role because the above-mentioned “connectedness property” of the greatest semilattice decomposition can be expressed as preservation of a very special kind of pullbacks (we shall make this observation explicit at the beginning of Section 5). Since pullbacks are in general not preserved, we will also look at some important special kinds of pullbacks as to their preservation. Several of them are not preserved either, nevertheless, we also get a positive result which is a far-reaching generalisation of the “connectedness property”. The special cases, however, do not figure in the table.

Some of the results hold in more general settings of universal algebra: for varieties with certain properties. However, instead of formulating results case by case in different settings, we prefer a uniform presentation for semigroups and semilattices. For a reader less familiar with universal algebra this approach may be easier to follow.

We are not going to define basic notions and to recall many fundamental facts from semigroup theory and category theory. Our general references on them are the books [3] and [5], for general properties of the greatest semilattice decomposition see [2] or [7].

2 Preliminaries

Let $S$ be a semigroup. By $S^1$ we denote the monoid obtained from $S$ by adjoining an identity if necessary. If $H \subseteq S \times S$, we denote by $\rho(H)$ the smallest semigroup congruence on $S$ containing the set $H$.

For a semigroup $S$ we also introduce the following notations:

$$C_S = \{(xy, yx) \mid x, y \in S\},$$

$$I_S = \{(x, x^2) \mid x \in S\}.$$
By $\delta_S$ we denote the smallest semilattice congruence on $S$, i.e., the congruence $\rho(C_S \cup I_S)$. We shall also consider the congruences $\alpha_S = \rho(C_S)$ and $\beta_S = \rho(I_S)$. Note that $\delta_S = \alpha_S \lor \beta_S$.

Recall that if $S$ is a commutative semigroup then $\delta_S = \beta_S$ and $s \delta_S t \iff (\exists x, y \in S)(\exists k, l \in \mathbb{N})(s^k = tx \land t^l = sy)$.

If $S$ is a band (that is, a semigroup in which every element is idempotent) then $\delta_S = \alpha_S$ and $s \delta_S t \iff s = sts \land t = tst$.

The assignment

$$
\begin{array}{ccc}
S & \rightarrow & S/\delta_S \\
\downarrow f & & \downarrow D(f) \\
T & \rightarrow & T/\delta_T
\end{array}
$$

where

$$
D(f)([s]_{\delta_S}) := [f(s)]_{\delta_T},
$$

defines a functor $D : \text{SGr} \rightarrow \text{SLat}$ to the category $\text{SLat}$ of semilattices, called the greatest semilattice image functor. Similarly, replacing $\delta$ by $\alpha$ or $\beta$ we obtain functors $C : \text{SGr} \rightarrow \text{CommSGr}$ to the category of commutative semigroups and $B : \text{SGr} \rightarrow \text{Band}$ to the category of bands. So we have a commutative diagram

$$
\begin{array}{ccc}
\text{SGr} & \rightarrow & \text{CommSGr} \\
C & \downarrow & D \\
\text{Band} & \rightarrow & \text{SLat}
\end{array}
\rightarrow
\begin{array}{ccc}
\text{SGr} & \rightarrow & \text{CommSGr} \\
B & \downarrow & D|_{\text{CommSGr}} \\
\text{Band} & \rightarrow & \text{SLat}
\end{array}
$$

of functors. In what follows, we often write just $D$ instead of $D|_{\text{CommSGr}}$ and $D|_{\text{Band}}$ if no confusion is possible. Since $D$ is a reflection, it preserves all colimits. Our aim in this paper is to study what kind of familiar limits are preserved by the functors in the above diagram, first of all by $D$, the most important of them.

We recall a classical result.

**Proposition 1** The following assertions are equivalent for a finitely complete category $C$ and a functor $F : C \rightarrow D$.

1. $F$ preserves all (finite) limits.
2. $F$ preserves (finite) products and pullbacks.
3. $F$ preserves (finite) products and equalizers.

We close the preliminaries by presenting a class of semigroups on which the functor $D$ behaves as well as possible: it preserves all limits.

Recall the following terminology from semigroup theory: if a semigroup $S$ admits a congruence $\theta$ such that $S/\theta$ is a semilattice and all congruence classes of $\theta$ belong to a class $\mathcal{P}$ of semigroups then we say that $S$ is a semilattice of semigroups from $\mathcal{P}$.

Recall further that a nil extension of a group is a semigroup $S$ with a subgroup $G$ such that every element of $S$ has a power in $G$. Denote by $\text{SLNEG}$ the full subcategory of $\text{SGr}$ whose objects are semilattices of nil extensions of groups.
Theorem 1 The functor \(D|_{\text{SLNEG}} : \text{SLNEG} \rightarrow \text{SLat}\) preserves all limits.

Proof. Notice that the mapping \(E\) which sends every semigroup to the set of its idempotents defines a functor since idempotents are preserved by homomorphisms. If we denote by \(U\) the forgetful functor \(\text{SLat} \rightarrow \text{Set}\) then the diagram

\[
\begin{array}{ccc}
\text{SLNEG} & \xrightarrow{D} & \text{SLat} \\
\downarrow{E} & & \downarrow{U} \\
\text{Set} & & 
\end{array}
\]

is commutative since every \(\delta\)-class of a semilattice of nil extensions of groups contains exactly one idempotent. Since the forgetful functor \(U\) creates limits, \(D\) preserves limits if and only if \(E\) does so, and the latter is true because, for every semigroup \(S\), \(E(S) \cong \text{Hom}(1, S)\) and the Hom functor preserves limits.

Remark 1 Theorem 1 speaks about the preservation of all existing limits in \(\text{SLNEG}\) but it says nothing about which limits actually exist. This question will be dealt with in a separate paper.

3 Products

First we consider the preservation of products.

Proposition 2 The functor \(B : \text{SGr} \rightarrow \text{Band}\) preserves finite products.

Proof. Let \(Q\) and \(R\) be arbitrary semigroups. Denote by \(\mathbb{N}\) the additive semigroup of positive integers. It is straightforward to see that \(\mathbb{N} \times \mathbb{N}\) is semilattice indecomposable (a consequence of \(\mathbb{N} \times \mathbb{N}\) being archimedean). Now, for any fixed \((q, r)\) in \(Q \times R\), sending \((k, l)\) to \((q^k, r^l)\) determines a homomorphism from \(\mathbb{N} \times \mathbb{N}\) to \(Q \times R\). Therefore, since \(\mathbb{N} \times \mathbb{N}\) is semilattice indecomposable, we must have \((q^k, r^l)\beta_{Q \times R}(q^m, r^n)\) for all \(q \in Q, r \in R, k, l, m, n \in \mathbb{N}\).

Now, fix any \(r \in R\) and consider the map \(Q \rightarrow B(Q \times R)\) which sends \(q\) to \([[(q, r)]\beta_{Q \times R}\) for every \(q \in Q\). By the foregoing, this map is a homomorphism. Therefore, by the universal property of \(B(Q)\), it induces a homomorphism \(B(Q) \rightarrow B(Q \times R)\). So \(q\beta_{Q}q'\) implies \((q, r)\beta_{Q \times R}(q', r)\), and similarly, for any \(q \in Q, r\beta_{R}r'\) implies \((q, r)\beta_{Q \times R}(q, r')\). This proves our claim.

Proposition 3 The functor \(D : \text{SGr} \rightarrow \text{SLat}\) preserves all products of unions of groups.

Proof. The proof follows from the facts that every semigroup which is a union of groups is a semilattice of completely simple semigroups, which are its \(\delta\)-classes, and the product of a family of completely simple semigroups is also completely simple.

Corollary 1 The functor \(D : \text{Band} \rightarrow \text{SLat}\) preserves all products.
From Proposition 2 and Corollary 1 we obtain

**Theorem 2** The functor $D : \text{SGr} \to \text{SLat}$ preserves finite products. ■

**Corollary 2** The (direct) product of two semilattice indecomposable semigroups is semilattice indecomposable. ■

**Example 1** The functor $C : \text{SGr} \to \text{CommSGr}$ does not preserve finite products.

Before presenting an example: in a semigroup $S$, denote by $S^2$ the subsemigroup consisting of those elements which can be written as a product, and by $\Delta_S$ the smallest (identical) congruence on $S$. Notice that $(S^2 \times S^2) \cup \Delta_S$ is a congruence which contains $\alpha_S$ because the latter congruence is generated by pairs of the form $(xy, yx)$. We will use this fact for a semigroup of the form $R \times R$.

$$\alpha_{R \times R} \subseteq ((R \times R)^2 \times (R \times R)^2) \cup \Delta_{R \times R}.$$ 

Take now any non-commutative semigroup $R$ with $R^2 \neq R$, and elements $r, s, t \in R$ with $rs \neq sr$ and $t \notin R^2$. Clearly, $((rs, t), (sr, t)) \notin ((R \times R)^2 \times (R \times R)^2) \cup \Delta_{R \times R}$, hence $(rs, t)$ and $(sr, t)$ give rise to distinct elements in $C(R \times R)$. On the other hand, they obviously give the same element in $C(R) \times C(R)$.

**Example 2** The functor $D : \text{CommSGr} \to \text{SLat}$ does not preserve all products. (This implies that $D : \text{SGr} \to \text{SLat}$ does not preserve all products, and $B : \text{SGr} \to \text{Band}$ does not preserve all products either since its restriction to $\text{CommSGr}$ coincides with the restriction of $D$.)

Denote by $\mathbb{N}$ the additive semigroup of positive integers, and let $Q$ be the infinite product $\mathbb{N} \times \mathbb{N} \times \ldots$. Observe that sending all bounded sequences to 1 and all unbounded sequences to 0 determines a surjective homomorphism from $Q$ to the two-element semilattice. Since such a homomorphism must factor through $D(Q)$, the latter has more than one element. On the other hand, $D(\mathbb{N})$ and hence also $D(\mathbb{N}) \times D(\mathbb{N}) \times \ldots$ is the one-element semilattice. Thus $D$ does not preserve the product $\mathbb{N} \times \mathbb{N} \times \ldots$.

**Remark 2** Ultraproducts. Filtered products (called also reduced products) generalize products and at the same time they are certain colimits of products. Since every left adjoint preserves all colimits, this means that any of our reflections preserves all filtered products if and only if it preserves all products. If we are interested in ultraproducts, their preservation of course follows from the preservation of products (because they are special filtered products) but the reflections $\text{SGr} \longrightarrow \text{SLat}, \text{SGr} \longrightarrow \text{Band}, \text{SGr} \longrightarrow \text{CommSGr}, \text{CommSGr} \longrightarrow \text{SLat}$ do not preserve products. To show that they do not preserve ultraproducts either, it suffices to exhibit this for $\text{SGr} \longrightarrow \text{CommSGr}$ and $\text{CommSGr} \longrightarrow \text{SLat}$. Examples 1 and 2, however, settle the ultraproduct case as well because certain elements (like $(1,2,3,\ldots)$ and $(1,1,1,\ldots)$) are non-equivalent not only in the original product but also in every product of a subfamily indexed by an infinite subset.

**Remark 3** Filtered limits. Any product $\prod_{i \in I} A_i$ is the filtered limit of the products over finite subsets of $I$. Therefore, whenever finite products are preserved and infinite ones not, we immediately conclude that filtered limits are not preserved. Accordingly, the reflections $\text{SGr} \longrightarrow \text{SLat}, \text{SGr} \longrightarrow \text{Band}, \text{CommSGr} \longrightarrow \text{SLat}$ do not preserve filtered limits. This is also true for $\text{SGr} \longrightarrow \text{CommSGr}$ because this reflection restricted to groups...
The functor Example 3 preserves finite products but not infinite products. (Of course, the same argument works for monoids.) Thus, it remains to look at the reflection Band \(\longrightarrow\) SLat.

Denote by \(N\) the set of natural numbers endowed with left zero multiplication (i.e., \(xy = x\) for all \(x, y \in N\)), and let \(N^1 = N \cup \{1\}\) where \(\{1\}\) is a formal identity element. Denote by \(D\) the diagram \(N^1 \leftarrow N^1 \leftarrow \ldots\), where each map sends \(1\) to \(1\) and, for all \(n \in N\), \(n\) to \(n + 1\). Then \(DD\) is \(L \leftarrow L \leftarrow \ldots\) where \(L\) is the two-element semilattice and every mapping is the identical mapping, hence \(\lim DD = L\), but \(\lim D\) and hence also \(D(\lim D)\) is the one-element semigroup.

4 Monomorphisms

Proposition 4 The functor \(D : \text{Band} \rightarrow \text{SLat}\) preserves all monomorphisms.

Proof. Let \(f : R \rightarrow S\) be a monomorphism of bands. We have to show that \(D(f) : R/\delta_R \rightarrow S/\delta_S\) is a monomorphism. Suppose that \(f(r)\delta_Sf(r')\), \(r, r' \in R\). Then \(f(r) = f(r)f(r')f(r)\) and \(f(r') = f(r')f(r)f(r')\) imply \(r = rr'r\) and \(r' = r'rr'\), which means that \(r\delta_Rr'\).

Example 3 The functor \(D : \text{CommSGr} \rightarrow \text{SLat}\) does not preserve all monomorphisms. (This implies that \(D : \text{SGr} \rightarrow \text{SLat}\) does not preserve all monomorphisms and, by Proposition 4, that \(B\) does not preserve all monomorphisms.)

Consider \(\mathbb{N}^1\), the additive semigroup of non-negative integers, and embed it into the group \(\mathbb{Z}\) of integers. Clearly, \(|D(\mathbb{N}^1)| = 2\) and \(|D(\mathbb{Z})| = 1\).

Still, \(D : \text{CommSGr} \rightarrow \text{SLat}\) preserves quite a big class of monomorphisms, namely relatively unitary monomorphisms, which turn up in the context of semigroup amalgams (see [3]).

A monomorphism \(f : R \rightarrow S\) of semigroups is said to be relatively unitary if \(f(R)\) is a relatively unitary subsemigroup of \(S\), that is,

\[
(\forall r \in R)(\forall s \in S)[f(r)s \in f(R) \Rightarrow (\exists r' \in R^1)(f(r)s = f(rr'))] \\
(\forall r \in R)(\forall s \in S)[sf(r) \in f(R) \Rightarrow (\exists r' \in R^1)(sf(r) = f(r'r))] 
\]

(where \(rr'\) with \(r \in R\) and \(r' \in R^1\) means \(r\) if \(r'\) is the formal identity element and \(rr'\) if \(r' \in R\)). Recall that a homomorphism \(f : R \rightarrow S\) of semigroups is called unitary, if either of \(f(r)s \in f(R)\) or \(sf(r) \in f(R)\) implies \(s \in f(R)\) for every \(r \in R\) and \(s \in S\). Clearly, unitary monomorphisms are relatively unitary, but there are also several other classes of monomorphisms strictly between unitary and relatively unitary ones, see Exercise 8.7(18) of [3].

Proposition 5 The functor \(D : \text{CommSGr} \rightarrow \text{SLat}\) preserves relatively unitary monomorphisms.

Proof. Let \(f : R \rightarrow S\) be a relatively unitary monomorphism of commutative semigroups. We have to show that \(f(r_1)\delta_Sf(r_2)\) implies \(r_1\delta_Rr_2\) for all \(r_1, r_2 \in R\). If \(f(r_1)\delta_Sf(r_2)\) then \(f(r_1^k) = f(r_1)^k = f(r_2)s_1\) and \(f(r_2^l) = f(r_2)^l = f(r_1)s_2\) for some \(k, l \in \mathbb{N}\) and \(s_1, s_2 \in S\). By the relative unitariness

\[
f(r_1^k) = f(r_2)s_1 = f(r_2) \quad \text{or} \quad f(r_1^k) = f(r_2)s_1 = f(r_2r_4), \\
f(r_2^l) = f(r_1)s_2 = f(r_1) \quad \text{or} \quad f(r_2^l) = f(r_1)s_2 = f(r_1r_3)
\]
for some \( r_3, r_4 \in R \). Hence \( r_1^k = r_2 \) or \( r_1^k = r_2 r_4 \), and \( r_2^l = r_1 \) or \( r_2^l = r_1 r_3 \). Consequently, \( r_1 \delta R r_2 \), which shows that \( D(f) : D(R) \to D(S) \) is a monomorphism.

To prove that \( D(f) \) is relatively unitary, suppose that

\[
D(f) ([r]_{\delta_R}) \cdot [s]_{\delta_S} \in D(f) (R/\delta_R),
\]

\( r \in R, s \in S \), that is, \( f(r) s \delta_S f(r_1) \) for some \( r_1 \in R \). Then \( f(r_1^l) = f(r) s s_1 \) for some \( l \in \mathbb{N} \) and \( s_1 \in S \). By the relative unitariness of \( f \), \( f(r) s s_1 = f(r r_2) \) for some \( r_2 \in R^l \). Hence

\[
[f(r)s]_{\delta_S} = [f(r_1)]_{\delta_S} = [f(r_1)]_{\delta_S}^l = [f(r_1^l)]_{\delta_S} = [f(r r_2)]_{\delta_S}.
\]

Thus \( D(f) ([r]_{\delta_R}) \cdot [s]_{\delta_S} = D(f) ([rr_2]_{\delta_R}) \), as needed. \( \blacksquare \)

The result of Proposition 5 does not extend to arbitrary semigroups.

**Example 4** The functor \( D : SGr \to SLat \) may take even unitary monomorphisms to morphisms which are not monomorphisms.

Let \( S = \langle x, y \rangle \) be the free semigroup on two generators, \( R = \langle xy, yx \rangle \) be the subsemigroup generated by the elements \( xy \) and \( yx \), and let \( f : R \to S \) be the inclusion. If \( rs \in R \) for \( r \in R, s \in S \), then both \( r \) and \( rs \) have to be products of \( xy \)'s and \( yx \)'s, hence also \( s \) has to be a product of these elements, i.e., \( s \in R \). Similarly, \( sr \in R \) implies \( s \in R \), and thus \( R \) is a unitary subsemigroup of \( S \), i.e., \( f \) is a unitary monomorphism. Obviously, \( xy \delta_S yx \). It is also clear that \( R \) is the free semigroup on \( \{xy, yx\} \), so \( D(R) \) is the free semilattice on the generators \([xy]_{\delta_R}\) and \([yx]_{\delta_R}\), hence \( (xy, yx) \notin \delta_R \). Thus \( D(f) \) is not a monomorphism.

**Example 5** The functor \( C : SGr \to CommSGr \) may take even unitary monomorphisms to morphisms which are not monomorphisms.

Embed any finite abelian group \( R \) into a symmetric group \( S \) on at least 2 elements. Then \( C(R) = R \) and \( C(S) \) is the two-element group.

5 Pullbacks and equalizers

Here we shall be concerned basically with pullback preservation; negative results about this will also yield negative results on equalizer preservation. Recall that a pullback is a commutative square

\[
\begin{array}{ccc}
P & \xrightarrow{p_2} & R \\
\downarrow{p_1} & & \downarrow{f} \\
Q & \xrightarrow{g} & S \\
\end{array}
\]
such that whenever $a_1: A \to Q$ and $a_2: A \to P$ satisfy $fa_2 = ga_1$, then there exists a unique $b: A \to P$ with $a_1 = p_1b$ and $a_2 = p_2b$.

In this case $P$ is determined by $f$ and $g$ up to isomorphism, and we write $P = Q \times_S R$. In our concrete case of semigroups we have $Q \times_S R = \{(q, r) \in Q \times R \mid g(q) = f(r)\}$.

Notice, first of all, that for a pullback

\[
\begin{array}{ccccccccc}
Q \times_S R & \to & R \\
\downarrow & & \downarrow \\
Q & \to & S
\end{array}
\]

there is always a canonical morphism $D(Q \times_S R) \to D(Q) \times_{D(S)} D(R)$, and the functor $D$ preserves this pullback if and only if this canonical morphism is an isomorphism.

Next, notice that the fact that the components of the greatest semilattice decomposition of a semigroup are themselves semilattice indecomposable (see [8]), can be expressed as a pullback preservation property: namely, preservation of the pullbacks of the form

\[
\begin{array}{ccccccccc}
S \times_{D(S)} 1 & \to & 1 \\
\downarrow & & \downarrow \\
S & \to & D(S)
\end{array}
\]  

Indeed, here $S \times_{D(S)} 1$ is exactly that class of the decomposition of $S$ which corresponds to the element $f(1)$ in $D(S)$, and preservation of this pullback means exactly that

\[D(S \times_{D(S)} 1) \cong D(S) \times_{D(S)} 1 \cong 1 ,\]

in other words, that this class is semilattice indecomposable.

Our next result shows that far more is true.

**Theorem 3** The functor $D : SGr \to SLat$ preserves all pullbacks

\[
\begin{array}{ccccccccc}
P & \to & R \\
\downarrow & & \downarrow \\
Q & \to & S
\end{array}
\]

in which $S$ is a semilattice.
Proof. Surjectivity of the canonical morphism $D(Q \times_S R) \to D(Q) \times_{D(S)} D(R)$ follows immediately from the fact that $D(S)$ is (canonically) the same as $S$: just observe that the surjective map $Q \times_S R \to D(Q) \times_{D(S)} D(R)$ factors through it. For the injectivity: the inverse image of any $([q], [r])$ in $D(Q) \times_{D(S)} D(R)$ under $Q \times_S R \to D(Q) \times_{D(S)} D(R)$ is precisely the product $[q] \times [r]$, and the latter is semilattice indecomposable by Corollary 2. Therefore the image of $[q] \times [r]$ under $Q \times_S R \to D(Q) \times_{D(S)} D(R)$ has only one element, and so $[q] \times [r]$ is contained in $[(q, r)]$. On the other hand, since the map $D(Q \times_S R) \to D(Q) \times_{D(S)} D(R)$ is well defined, $[(q, r)]$ is contained in $[q] \times [r]$. That is, $[(q, r)]$ is the same as the inverse image of $([q], [r])$ in $Q \times_S R$, and so the map $D(Q \times_S R) \to D(Q) \times_{D(S)} D(R)$ is injective. $lacksquare$

Remark 4 Theorem 3 says exactly that the reflection of semigroups to semilattices has stable units in the sense of [1].

Example 6 The functor $C : SGr \to \mathbf{CommSGr}$ does not preserve all pullbacks (2) where $S$ is a commutative semigroup, because the product $R \times R$ in Example 1 is a pullback of this type (with $S = \{1\}$) and it is not preserved. Also, by a similar argument it follows from Example 2 that $D : SGr \to \mathbf{SLat}$ does not preserve all multiple pullbacks in which the lower right corner is a semilattice.

Example 7 The functor $B : SGr \to \mathbf{Band}$ does not preserve all pullbacks (2) with $S$ a band. In fact, by Tamura[9], the greatest band decomposition of a semigroup may have classes which admit non-trivial band decomposition, and this means that there exist pullbacks of the form

$$
\begin{array}{ccc}
S \times_{B(S)} 1 & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
S & \longrightarrow & B(S)
\end{array}
$$

which are not preserved by $B$.

Before giving the next positive result, we advance a statement which may be of independent interest.

Proposition 6 For the restriction of $D$ to $\mathbf{Band}$, the canonical morphism $D(Q \times_S R) \to D(Q) \times_{D(S)} D(R)$ is always injective.

Proof. This follows immediately from the fact that two elements $r, s$ of a band belong to the same $\delta$-class if and only if they satisfy the identities $r s r = r$ and $s r s = s$. $lacksquare$

Proposition 7 The functor $D : \mathbf{Band} \to \mathbf{SLat}$ preserves all pullbacks

$$
\begin{array}{ccc}
Q \times_S R & \longrightarrow & R \\
\downarrow & & \downarrow f \\
Q & \longrightarrow & S
\end{array}
$$

such that $f$ is a monomorphism and $g(Q) \cap \overline{f(R)} \subseteq f(R)$, where $\overline{f(R)}$ denotes the union of those $\delta_S$-classes which intersect $f(R)$.
Proof. Since the canonical map $D(Q \times_S R) \longrightarrow D(Q) \times_{D(S)} D(R)$ is injective by Proposition 6, it remains to prove only its surjectivity. Take any element $([q]_{\delta Q}, [r]_{\delta R})$ from $D(Q) \times_{D(S)} D(R)$, then we have $g(q)f(r)g(q) = g(q)$ and $f(r)g(q)f(r) = f(r)$ since $g(q)\delta_S f(r)$. By the assumption there is an $r' \in R$ such that $g(q) = f(r')$, so that $f(r')f(r)f(r') = f(r')$ and $f(r)f(r')f(r) = f(r)$. By the injectivity of $f$, we get $r'r'r' = r'$ and $rr'r = r$, that is, $r\delta_R r'$. Thus we have $(q, r') \in Q \times_SR$ and $([q]_{\delta Q}, [r']_{\delta R}) = ([q]_{\delta Q}, [r]_{\delta R})$, as required. ■

Remark 5 Proposition 7 says that $D$ preserves inverse images of subbands under a large class of morphisms. In particular, if $R$ is a subband of $S$ and $g : Q \longrightarrow S$ is an arbitrary homomorphism of bands such that either $R$ is a union of classes of the finest semilattice decomposition of $S$ or $g(Q) \subseteq R$ then $D$ preserves the inverse image of $R$ under $g$. Notice that the last case yields the statement of Proposition 4 about the preservation of band monomorphisms: we take $f = g$ in Proposition 7.

In addition to the preservation of the abovementioned pullbacks, by applying categorical Galois theory to the semi-left-exact reflection $D \dashv I : \text{SLat} \xrightarrow{\text{Band}}$ (where $\text{I} : \text{SLat} \longrightarrow \text{Band}$ is the inclusion functor) one can conclude that $D|_{\text{Band}}$ preserves also some other very specific pullbacks (see [4]). Namely if in the diagram

$$
P \xrightarrow{p_2} R \xrightarrow{f} R/\delta_R (= D|_{\text{Band}}(R))$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$Q \xrightarrow{g} S \xrightarrow{D|_{\text{Band}}(f)} S/\delta_S (= D|_{\text{Band}}(S))$$

both squares are pullbacks and $D|_{\text{Band}}(g)$ is an isomorphism then also $D|_{\text{Band}}(p_1)$ is an isomorphism and $D|_{\text{Band}}$ preserves the left-hand side pullback.

Also (see again [4]), $D|_{\text{Band}}$ preserves the kernel pairs of band homomorphisms $\tau : S \rightarrow T$ such that $\alpha_S \circ \ker \tau = \ker \tau \circ \alpha_S$.

By now we have seen that the functors $C : \text{SGr} \rightarrow \text{CommSGr}$ and $B : \text{SGr} \rightarrow \text{Band}$ do not preserve all pullbacks (see Examples 6 and 7). With the subsequent examples we show not only that pullbacks, in general, are not preserved by $D$, but also that even such important special pullbacks as kernel pairs of surjective homomorphisms from free semigroups (bands, commutative semigroups), kernel pairs of split epimorphisms, or kernel pairs of surjective endomorphisms need not be preserved.

For constructing counterexamples, notice that if $D : \text{SGr} \rightarrow \text{SLat}$ preserves a pullback

$$
Q \times_S R \xrightarrow{p_2} R \xrightarrow{f} R\xrightarrow{p_1} Q \xrightarrow{g} S,
$$

then the canonical morphism $D(Q \times_S R) \longrightarrow D(Q) \times_{D(S)} D(R)$ must be surjective, i.e., for every $r \in R$ and $q \in Q$,

$$
if \, g(q)\delta_S f(r) \text{ then there exist } r' \in R \text{ and } q' \in Q \text{ such that } g(q') = f(r'), \, q'\delta_Q q \text{ and } r'\delta_R r'.
$$
Most of our counterexamples will amount to presenting elements \( q \in Q \) and \( r \in R \) such that \( g(q)\delta_S f(r) \) but there are no \( q' \in Q \) and \( r' \in R \) as required in condition (5).

**Example 8** The functor \( D : SGr \to SLat \) does not preserve kernel pairs of surjective homomorphisms \( f : R \to S \), where \( R \) is a free semigroup. Consequently, \( D : SGr \to SLat \) does not preserve pullbacks.

Let \( S = \{a, b\} \) be a left zero semigroup and \( R = \langle x, y \rangle \) the free semigroup on two generators. We define a homomorphism \( f : R \to S \) by

\[
\begin{align*}
    f(x) & := a, \\
    f(y) & := b.
\end{align*}
\]

Then \( f(x)\delta_S f(y) \). Note that two words from \( R \) are \( \delta_R \)-related if and only if they consist of the same letters. Hence we cannot find \( r', q' \in R \) such that \( f(r') = f(q') \), \( r'\delta_R x \), and \( q'\delta_R y \). Since condition (5) is not fulfilled, \( D \) cannot preserve the kernel pair of \( f \).

**Example 9** The functor \( D : Band \to SLat \) does not preserve kernel pairs of surjective homomorphisms \( f : R \to S \), where \( R \) is a free band. Consequently, \( D : Band \to SLat \) does not preserve pullbacks.

Let \( S = \{a, b\} \) be a left zero semigroup and \( R' = \langle x, y \rangle \) the free band on two generators, and define \( f' : R' \to S \) similarly to \( f \) in Example 8. Now the map \( f : R \to S \) from Example 8 can be written as \( f = f'g \), and \( R \to D(R) \) factors through the same \( g \), hence we have the same situation here as in Example 8.

**Example 10** The functor \( D : CommSGr \to SLat \) does not preserve kernel pairs of surjective homomorphisms \( f : R \to S \), where \( R \) is a free commutative semigroup. Consequently, \( D : CommSGr \to SLat \) does not preserve pullbacks.

Let \( \mathbb{Z} \) be the additive group of integers, \( R = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid m \geq 0, n \geq 0, m + n > 0\} \) (which is a free semigroup on \( \{(1, 0), (0, 1)\} \)) and let \( f : R \to \mathbb{Z} \) denote the homomorphism defined by \( f(m, n) = m - n \). We observe:

(a) Since \( D(\mathbb{Z}) \) has only one element, we have \( f(1, 0)\delta_\mathbb{Z} f(0, 1) \).

(b) Suppose \( (m, n)\delta_R (1, 0) \) and \( (p, q)\delta_R (0, 1) \). Then, according to the obvious description of \( D(R) \), we have \( n = 0 = p \). Since the numbers \( m \) and \( q \) must then be positive, we conclude that \( f(m, n) \) must be different from \( f(p, q) \).

(c) As follows from condition (5), (a) and (b) together tell us that \( D \) does not preserve the pullback of \( f \) along itself.

**Example 11** The functor \( D : Band \to SLat \) does not preserve kernel pairs of split epimorphisms (hence the same holds for \( D : SGr \to SLat \), too).

Consider the band \( R = \{a, b, r, q\} \) with the multiplication table

\[
\begin{array}{c|cccc}
  & a & b & r & q \\
\hline
a & a & a & a & a \\
b & b & b & b & b \\
r & a & a & r & a \\
q & b & b & b & q
\end{array}
\]

and its subband \( S = \{a, b\} \). Note that \( R \) is a semilattice of left zero semigroups \( \{a, b\}, \{r\} \) and \( \{q\} \), where \( \{a, b\} \) is the smallest component and \( \{r\}, \{q\} \) are incomparable. Let homomorphisms \( f : R \to S \) and \( g : S \to R \) be given by the tables

- for \( f(x) \)
  \[
  \begin{array}{c|cccc}
   x & a & b & r & q \\
\hline
   f(x) & a & b & a & b
  \end{array}
  \]

- for \( g(x) \)
  \[
  \begin{array}{c|cc}
   x & a & b \\
\hline
   g(x) & a & b
  \end{array}
  \]
Then \( fg = 1_S \), so \( f \) is a split epimorphism and \( S \) is a retract of \( R \). Now \( f(r) = a \alpha_S b = f(q) \), but \( f(r) \neq f(q) \) and \( r \) and \( q \) are the only elements in their equivalence classes, hence condition (5) is not satisfied, so \( D \) cannot preserve the kernel pair of \( f \).

Notice that the statement of Example 11 remains valid if we replace ‘split epimorphism’ by ‘surjective endomorphism’. Indeed, if we have a split-epi-counterexample \( f : R \rightarrow S \) then take the coproduct \( A = S + R + R + \ldots \) and define \( h : A \rightarrow A \) as the homomorphism induced by the sequence \((i_1, i_1 f, i_2, i_3, \ldots)\), where \( i_1 : S \rightarrow A \) and \( i_n : R \rightarrow A \), \( (n = 2, 3, \ldots) \), are the coproduct injections. It is straightforward to verify that the kernel pair of \( h \) is not preserved by \( D \) – details are omitted here.

With Proposition 2, Corollary 1, Theorem 2 and the examples in the present section we have also settled the question of equalizer preservation. Namely, the functors \( D \) and \( B \) preserve finite products but not pullbacks, hence they cannot preserve equalizers in view of Proposition 1. Since even the restrictions of \( D \) to \( \text{CommSGr} \) and to \( \text{Band} \) do not preserve pullbacks, they cannot preserve equalizers either. But if \( D : \text{Band} \rightarrow \text{SLat} \) does not preserve equalizers then the same holds for the functor \( C \) as well, because \( \delta_S = \alpha_S \) for every band \( S \).

6 Conclusion for semigroups

We summarize the validity of the main preservation properties in the following table.

|                | \( D \) | \( D|_{\text{Band}} \) | \( D|_{\text{CommSGr}} \) | \( B \) | \( C \) |
|----------------|--------|------------------------|------------------------|--------|--------|
| finite products| yes    | yes                    | yes                    | yes    | no     |
|                | Thm. 2 | Cor. 1                 | Thm. 2                 | Prop. 2| Ex. 1  |
| products       | no     | yes                    | no                     | no     | no     |
|                | Ex. 2  | Cor. 1                 | Ex. 2                  | Ex. 2  | Ex. 1  |
| monomorphisms  | no     | yes                    | no                     | no     | no     |
|                | Ex. 3  | Prop. 4                | Ex. 3                  | Ex. 3  | Ex. 5  |
| pullbacks      | no     | no                     | no                     | no     | no     |
|                | Ex. 8  | Ex. 9                  | Ex. 10                 | Ex. 7  | Ex. 6  |
| equalizers     | no     | no                     | no                     | no     | no     |

7 Groups and monoids

Since, for a group \( G \), obviously \(|B(G)| = 1 = |D(G)|\), our discussion on the preservation of limits by \( B \) and \( D \) becomes trivial in the case of groups (that is, all limits of groups are trivially preserved). The situation with the functor \( C \) is also very simple, but quite different. Indeed, if \( G \) is a group, then \( C(G) = G/[G, G] = H_1(G, \mathbb{Z}) \) (= the first homology group of \( G \) with coefficients in the additive group of integers considered as a trivial \( G \)-module). And, as it is well known in homological algebra/group theory, \( H_1(\cdot, \mathbb{Z}) \) preserves finite products but not other limits in general. Moreover, a reader familiar with elementary group theory will immediately observe:

(a) Preservation of finite products and non-preservation of infinite ones: Since \([G, G]\) above consists of products of commutators of elements in \( G \), and, for \( m < n \), any product of \( m \) commutators can be presented as a product of \( n \) commutators, we can write \([G \times G', G \times G'] = [G, G] \times [G', G']\), and so \( G \times G'/[G \times G', G \times G'] = G/[G, G] \times G'/[G', G']\). On
the other hand, an infinite product might involve a sequence of products of commutators with an unbounded sequence \( m < n < \ldots \) instead of the two numbers \( m \) and \( n \) above; therefore not all infinite products are preserved.

(b) Consider the short exact sequence

\[
0 \rightarrow A_3 \rightarrow S_3 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0,
\]

where \( A_3 = \mathbb{Z}/3\mathbb{Z} \) and \( S_3 \) are the alternating and the symmetric group (on 3 elements), respectively. The image of this sequence under the functor \( C \) is

\[
0 \rightarrow \mathbb{Z}/3\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \rightarrow 0,
\]
giving a counterexample for the preservation of monomorphisms (as in our Example 5), but also a counterexample for the preservation of kernels, and hence a counterexample for the preservation of equalizers.

It is easy to check that all our conclusions for semigroups hold also for monoids (again, we skip the details), with one exception: in the monoid case, the functor \( C \) does preserve finite products. As a matter of fact, any reflection between varieties of monoids does so: indeed, given \( a, a' \) in a monoid \( A \) and \( b, b' \) in a monoid \( B \) (in the larger variety) we have

\[
a \sim a' \land b \sim b' \Rightarrow (a, 1) \sim (a', 1) \land (1, b) \sim (1, b') \Rightarrow (a, b) \sim (a', b')
\]

where \( \sim \) denotes the congruence induced by the given reflection on the considered monoid.

References


[4] V. Laan, Categorical Galois theory applied to the greatest semilattice decomposition of bands, manuscript.


