IMPULSIVE PERTURBATIONS OF A THREE-TROPHIC PREY-DEPENDENT FOOD CHAIN SYSTEM

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Abstract. The dynamics of an impulsively controlled three-trophic food chain system with general nonlinear functional responses for the intermediate consumer and the top predator is analyzed using Floquet theory and comparison techniques. It is assumed that the impulsive controls act in a periodic fashion, the constant impulse (the biological control) and the proportional impulses (the chemical controls) acting with the same period, but at different moments. Sufficient conditions for the global stability of resource and intermediate consumer-free periodic solution and of the intermediate consumer-free periodic solution are established, the latter corresponding to the success of the integrated pest management strategy from which our food chain system arises. In this regard, it is seen that, theoretically speaking, the control strategy can be always made to succeed globally if proper pesticides are employed, while as far as the biological control is concerned, its global effectiveness can also be reached provided that the top predator is voracious enough or the (constant) amount of top predators released each time is large enough or the release period is small enough. Some situations which lead to chaotic behavior of the system are also investigated by means of numerical simulations.

Key words. Simple food chain, integrated pest management, impulsive perturbations, global stability, chaos.

AMS subject classifications. 92D40, 92D25, 34A37, 34D23

1. Introduction. Classical two species continuous time models have constituted for a long time the main tool used to investigate the interactions between ecological populations (see, for instance, Volterra [23], Leslie-Gower [13], May [16]). However, as seen from Poincaré’s theorem, such models have only two behavior patterns, that is, they approach either a limit cycle or an equilibrium point and consequently fail to capture the complex behavior of some natural ecosystems. Further, other shortcomings of certain two-species models have also been pointed out. These are the paradox of enrichment (Rosenzweig [20]), which states that an increase in the carrying capacity of the environment in a Lotka-Volterra model will cause an increase in the size of the predator class at equilibrium, but not in that of the prey class, and the paradox of biological control (Luck [15]), which states that the low prey equilibrium densities of a Lotka-Volterra model are inherently unstable. Other authors have also brought criticism upon ratio-dependent type models, which were introduced as a replacement of Lotka-Volterra models (see Abrams [1], Deng et al [4]).

Consequently, another paradigm started to prevail, that is, the idea that the behavior of a complex system can be understood only when mutual interactions between a larger number of species are considered in a single model (Rosenzweig [21]). As a result, complex behavior, in the form of stable equilibria, limit cycles, multiple attractors and chaos, has been observed in three or more species models (Gilpin [6]) and it has also been noted that the dynamical outcome may depend on the initial population sizes, which is more in line with the results of field experiments and observations.

To understand the dynamical behavior of ecological communities, one should start by tracing their food webs and quantifying the strength of the respective interspecies

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interactions. It has been observed by Hastings and Powell in [7] and by Klebanoff and Hastings in [11] that since food webs often describe a net of nonlinear predator-prey interactions, there is a natural tendency of food webs to oscillate and chaos may ultimately arise when two or more predator-prey subsystems oscillate with incommensurable frequencies. McCann and Yodzis [18] mention that the parameter values chosen by Hastings and Powell in [7] may be biologically unfeasible, but the conclusions obtained in [7] are valid, and indicate biologically reasonable sets of parameter values which also produce chaos. They also provide comments about which biological conditions (metabolic types) favor the apparition of chaos. Six natural types of food web configurations are studied in McCann et al [17] and it is also found that the dominance of strong consumer-resource interactions may generate cyclic dynamics when the frequencies of oscillation are commensurate, respectively chaotic dynamics when the frequencies of oscillation are incommensurate, while the dominance of weak coupling between interactions may dampen the total oscillation of the system, together with other biological factors, such as omnivory and food-chain-predation mechanisms. See also Bascompte et al [3].

The so-called simple food chain, which is studied in our paper, is a tritrophic food chain which appears when a top predator $P$ feeds on an intermediate consumer $C$, which in turn feeds on a resource $R$. In this model, neither the intermediate consumer nor the top predator feed on other resources and nutrient recycling is not accounted for. The qualitative behavior of the simple food chain model with Holling type II functional responses for both the top predator and the intermediate consumer, that is, for $g_i(x) = \frac{\alpha_i x}{1+\beta_i x}$, $i \in \{1, 2\}$, has been studied in detail by Hastings and Powell in [7] and by Klebanoff and Hastings in [11]. See also McCann and Yodzis [19], Kuznetsov and Rinaldi [12]. In these papers, it has been found that the model may exhibit chaotic behaviour in the neighborhood of the intermediate consumer-free equilibrium and it has also been observed that the half saturation rate of the intermediate consumer $b_1$ is a key parameter for the stability of the model. A thorough analysis of the simple food chain model with ratio-dependent functional response for both the top predator and the intermediate consumer has been performed in Hsu et al [9]. Particularly, a tristability situation has been observed, in which different solutions tend to the origin, intermediate consumer-free equilibrium and positive equilibrium, respectively, for the same set of parameters and a discussion of the feasibility of the biological control has also been provided. Chaotic-looking solutions have also been found to exist for certain parameter values.

Simple food chain models may naturally appear as a result of integrated pest management strategies. As it has been noted that the abuse of pesticides has undesirable long-term environmental consequences, in order to regulate pest populations use is often made of different methods which are specifically suited to the target pest and minimize the harmful effects on the environment or on non-target organisms.

Biological control is defined as the reduction of pest populations by using their natural enemies (see Hoffmann and Frodsham [8]). While an approach to biological control (importation) relies on the import of exotic natural enemies, other approaches (augmentation and conservation) rely on supplementing or manipulating the existing natural enemies in order to enhance their effectiveness and on modifying the environment, respectively. A way to achieve augmentation is to release pest pathogens or infected pests with the purpose of generating an epidemics in the pest population, on the grounds that infected pests generally cause less environmental damage, another one being to breed natural predators of the pest in laboratories and to release them
periodically in the ecosystem. Consequently, in our food chain model \( R \) is the resource to be protected, \( C \) is the pest which should be regulates and \( P \) is a natural predator of the pest which is augmented by means of periodic release of laboratory-bred individuals.

Adequate efficiency is attained when at least one of these approaches is combined with the responsible use of chemical controls (pesticide spraying) and the use of mechanical accessories, such as pest barriers and pest traps, in the form of an integrated pest management strategy. Note that the integrated pest management strategy is considered successful when the pest population is reduced under certain economically significant levels, rather than when the pest population is totally eradicated, as the latter might be economically or logistically unfeasible, or it might be potentially damaging to the environment. In this regard, the economic injury level (EIL) is defined in Stern et al [22] as the amount of pest injury which will justify the cost of using controls or the lowest pest density which causes economic damage.

Due to the inherent discontinuity of human activities (that is, pesticides cannot be sprayed all year round but only during certain periods of the year), a natural choice is to use discrete impulsive controls rather than continuous controls for our pest management strategy. In this regard, the effect of impulsive perturbations on the simple food chain model has been studied by Zhang and Chen [24] assuming linear responses for the top predator and the intermediate consumer, by Zhang and Chen in [25] assuming Holling type II functional responses, by Zhang et al in [29] assuming Holling type IV (or simplified Monod-Haldane) functional responses and by Zhang et al in [28] assuming Beddington-DeAngelis functional responses. In all these papers, only the case of a constant impulsive perturbation has been considered. See also Zhang et al [27], Zhang et al [30], Liu et al [14] for related results regarding the impulsive control of predator-prey systems and Georgescu and Moresanu [5] for the discussion of an integrated pest management strategy involving biological and chemical impulsive controls.

2. The model. The abundance and interaction of resource, intermediate consumer and top predator populations may be expressed in terms of their biomass per spatial unit. In this regard, let \( x(t), y(t), z(t) \) be the biomass per spatial unit of the resource, intermediate consumer and top predator, respectively. As previously mentioned, we assume that the top predator feeds on the intermediate consumer only and in turn the intermediate consumer feeds on the resource only, while the nutrient recycling is not accounted for.

The functional responses of the intermediate consumer and of the top predator are denoted by the nonlinear smooth functions \( g_1, g_2 \), depending only on the resource biomass density and on the intermediate consumer biomass density, respectively, and satisfying a few assumptions which will be outlined below. Due to the assumption above, our model may be called, following the terminology given in Huisman and DeBoer [10], prey dependent, as opposed to a model in which the functional response of the predators are functions of the prey-to-predator ratios, which is called ratio-dependent (or, more generally, predator-dependent). It is supposed that in the absence of predation from the intermediate consumer, the resource grows according to a logistic growth with intrinsic growth rate \( r \) and carrying capacity \( r/a \).

The processes of resource conversion into intermediate consumer biomass and of intermediate consumer into top predator biomass, respectively, are characterized by constant conversion rates \( k_1 \) and \( k_2 \). The death rates \( d_1 \) and \( d_2 \) of the intermediate consumer and of the top predator, respectively, are also assumed to be constant.
It is assumed that top predators are bred in laboratories and subsequently released in an impulsive and periodic fashion of period $T$, in a fixed amount $\mu$ each time. It is also assumed that pesticides are sprayed in an impulsive and periodic fashion, with the same period as the action of releasing top predators, but at different moments. As a result of pesticide spraying, fixed proportions $\delta_1, \delta_2, \delta_3$ of the resource, intermediate consumer and top predator biomass are degraded each time.

On the basis of the above assumptions, we may formulate the following impulsively perturbed model

\[
\begin{align*}
    x'(t) &= x(t)(r - ax(t)) - g_1(x(t))y(t), & t \neq (n + l - 1)T, t \neq nT; \\
    y'(t) &= k_1g_1(x(t))y(t) - g_2(y(t))z(t) - d_1y(t), & t \neq (n + l - 1)T, t \neq nT; \\
    z'(t) &= k_2g_2(y(t))z(t) - d_2z(t), & t \neq (n + l - 1)T, t \neq nT; \\
    \Delta x(t) &= -\delta_1x(t), & t = (n + l - 1)T; \\
    \Delta y(t) &= -\delta_2y(t), & t = (n + l - 1)T; \\
    \Delta z(t) &= -\delta_3z(t), & t = (n + l - 1)T; \\
    \Delta x(t) &= 0, & t = nT; \\
    \Delta y(t) &= 0, & t = nT; \\
    \Delta z(t) &= \mu, & t = nT.
\end{align*}
\]

Here, $T > 0$, $0 < l < 1$, $\Delta \varphi(t) = \varphi(t+) - \varphi(t)$ for $\varphi \in \{x, y, z\}$ and $t > 0$, $0 \leq \delta_1, \delta_2, \delta_3 < 1$, $n \in \mathbb{N}^*$. The functions $g_1, g_2$ are assumed to satisfy the following assumptions.

\begin{enumerate}
    \item[(G)] $g_i$ is of class $C^1$ on $\mathbb{R}_+$, $g_i(0) = 0$, increasing and such that $x \mapsto g_i(x)/x$ is decreasing on $\mathbb{R}_+$, $|g'_i(x)| \leq L_i$ for $x \in \mathbb{R}_+$, $i \in \{1, 2\}$, where $L_1, L_2 \geq 0$.
\end{enumerate}

Note that hypothesis (G) is satisfied if functions $g_1, g_2$ represent Holling type II functional responses, that is, $g_i(x) = \frac{a_i x}{1 + b_i x}$, $i \in \{1, 2\}$, in which $a_i$, $i \in \{1, 2\}$ are the search rates of the resource and of the intermediate consumer, respectively, and $b_i$, $i \in \{1, 2\}$, are the corresponding half-saturation constants, that is, the resource (respectively intermediate consumer) biomass level at which the predation rate per unit resource (respectively per intermediate consumer) is half of its maximum value. Also, the above-mentioned constants $L_1$ and $L_2$ can be taken as globally Lipschitz constants for $g_1, g_2$, respectively.

Impulsive perturbations of our three trophic food chain model have also been considered by Zhang and Chen in [25], in the form of the periodic constant impulsive perturbations of the top predator only (that is, no second group of conditions in (S)), with particular Holling type II functional responses for the intermediate consumer and for the top predator. In [25], the local asymptotic stability of the intermediate consumer-extinction periodic solution is established, provided that the impulsive period $T$ is small enough, and it is also shown that the resource and intermediate consumer-free periodic solution is unstable.

By using similar techniques, that is, Floquet theory of impulsively perturbed systems of ordinary differential equations and comparison techniques, we are also able to prove further global stability results for both the intermediate consumer-free periodic solution and the resource and intermediate consumer-free periodic solution, under appropriate conditions, the former result representing a sufficient condition for the success of our pest control strategy. Note that, due to the impulsive top predator release of constant strength, our controlled system does not exhibit the domino effect, characteristic to the unperturbed food chain system, that is, if one species dies out.
then all the species at higher trophic levels die out as well (although the extinction of the resource will attract the extinction of the intermediate consumer, of course). Due to the proportional impulsive perturbations at \( t = (n + l - 1)T \), \( n \in \mathbb{N}^* \), the resource and intermediate consumer-free periodic solution is no longer unstable for any values the parameters involved, as it is the case when only constant impulsive perturbations of \( z \) are employed, and the existence of a threshold parameter which controls its stability is also established. Also, our food chain system may be interpreted as the nonlinear coupling of two predator-prey subsystems (intermediate consumer-resource and top predator-intermediate consumer) through the mediation of the intermediate consumer, while the impulsive perturbations induce commensurate oscillations, as they act with the same period \( T \). It is therefore expected that the system will display an oscillatory behavior, tending to a (impulsively perturbed) limit cycle of period \( T \) for an important portion of the parameter space, corresponding to impulsive and periodic perturbations with significant strength.

3. Preliminaries. In this section we shall introduce a few definitions and notations together with a few auxiliary results relating to comparison methods and Floquet theory for impulsively perturbed systems of ordinary differential equations. The biological well-posedness of the Cauchy problem associated to our system (S) for strictly positive initial data will also be established.

Let us denote by \( f = (f_1, f_2, f_3) \) the mapping defined by the right-hand sides of the first three equations in (S). Let also \( \mathcal{V}_0 \) be the set of functions \( V : \mathbb{R}_+ \times \mathbb{R}_+^3 \to \mathbb{R}_+ \) which are locally Lipschitz in the second variable, continuous on \( ((n + l - 1)T, nT] \times \mathbb{R}_+^3 \) and on \((nT, (n + l)T] \times \mathbb{R}_+^3 \) and for which the limits \( \lim_{(t, y) \to ((n + l - 1)T+, x)} V(t, y) = V((n + l - 1)T+, x) \) and \( \lim_{(t, y) \to (nT+, x)} V(t, y) = V(nT+, x) \) exist and are finite for \( x \in \mathbb{R}_+^3 \) and \( n \in \mathbb{N}^* \).

For \( V \in \mathcal{V}_0 \), we define the upper right Dini derivative of \( V \) with respect to the system (S) at \((t, x) \in ((n + l - 1)T, nT] \times \mathbb{R}_+^3 \) or \((nT, (n + l)T] \times \mathbb{R}_+^3 \) by

\[
D^+ V(t, x) = \lim_{h \downarrow 0} \frac{1}{h} [V(t + h, x + hf(t, x)) - V(t, x)].
\]

We now indicate a comparison result for solutions of impulsive differential inequalities which allows us to estimate the values of the solutions of (S). We suppose that \( h : \mathbb{R}_+ \to \mathbb{R} \) satisfies the following hypotheses.

(H) \( h \) is continuous on \((n + l - 1)T, nT] \times \mathbb{R}_+ \) and on \((nT, (n + l)T] \times \mathbb{R}_+ \) and the limits \( \lim_{(t, y) \to ((n + l - 1)T+, x)} h(t, y) = h((n + l - 1)T+, x) \), \( \lim_{(t, y) \to (nT+, x)} h(t, y) = h(nT+, x) \) exist and are finite for \( x \in \mathbb{R}_+^3 \) and \( n \in \mathbb{N}^* \).

Lemma 3.1. ([2]) Let \( V \in \mathcal{V}_0 \) and assume that

\[
\begin{align*}
D^+ V(t, x(t)) &\leq h(t, V(t, x(t))), & t &\neq (n + l - 1)T, nT; \\
V(t, x(t+)) &\leq \psi^n_1(V(t, x(t))), & t &= (n + l - 1)T; \\
V(t, x(t+)) &\leq \psi^n_2(V(t, x(t))), & t &= nT.
\end{align*}
\]

where \( h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \) satisfies (H) and \( \psi^n_1, \psi^n_2 : \mathbb{R}_+ \to \mathbb{R}_+ \) are nondecreasing for
all \( n \in \mathbb{N}^* \). Let \( r(t) \) be the maximal solution of the impulsive Cauchy problem

\[
\begin{aligned}
   u'(t) &= h(t, u(t)), \quad t \neq (n + l - 1)T, nT; \\
   u(t+) &= \psi^1_n(u(t)), \quad t = (n + l - 1)T; \\
   u(t+) &= \psi^2_n(u(t)), \quad t = nT; \\
   u(0+) &= u_0
\end{aligned}
\]

(3.2)

defined on \([0, \infty)\). Then \( V(0+, x_0) \leq u_0 \) implies that \( V(t, x(t)) \leq r(t) \) for all \( t \geq 0 \), where \( x(t) \) is an arbitrary solution of (3.1).

Note that under appropriate regularity conditions the Cauchy problem (3.2) has a unique solution and in that case \( r \) becomes the unique solution of (3.2). We now indicate a result which provides estimations for the solution of a system of differential inequalities.

**Lemma 3.2.** ([2]) Let the function \( u \in PC^4(\mathbb{R}_+, \mathbb{R}) \) satisfy the inequalities

\[
\begin{aligned}
   \frac{du}{dt} &\leq (\geq) p(t)u(t) + f(t), \quad t \neq \tau_k, \ t > 0; \\
   u(\tau_k+) &\leq (\geq) d_k u(\tau_k) + h_k, \ k \geq 0; \\
   u(0+) &\leq (\geq) u_0,
\end{aligned}
\]

(3.3)

where \( p, f \in PC(\mathbb{R}_+, \mathbb{R}) \) and \( d_k \geq 0 \), \( h_k \) and \( u_0 \) are constants and \( (\tau_k)_{k \geq 0} \) is a strictly increasing sequence of positive real numbers. Then, for \( t > 0 \),

\[
   u(t) \leq (\geq) u_0 \left( \prod_{0 < \tau_k < t} d_k \right) e^{\int_0^t \prod_{0 \leq \tau_k < t} d_k} \int_0^t \left( \prod_{0 \leq \tau_k < t} d_k \right) e^{\int_0^t \prod_{0 \leq \tau_k < t} d_k} f(s)ds + \sum_{0 < \tau_k < t} \left( \prod_{\tau_k < \tau_j < t} d_j \right) e^{\int_0^t \prod_{\tau_k < \tau_j < t} d_j} h_k.
\]

In the above, by \( PC(\mathbb{R}_+, \mathbb{R}) \) (\( PC^4(\mathbb{R}_+, \mathbb{R}) \)) is meant the class of real piecewise continuous (real piecewise continuously differentiable) functions defined on \( \mathbb{R}_+ \). For other results on impulsive differential equations, see Bainov and Simeonov [2]. Using Lemma 3.2, it is now possible to prove that the Cauchy problem with strictly positive initial data is well-posed for our system (S), that is, solutions \( (x, y, z) \) starting with strictly positive initial data remain strictly positive and bounded on their whole domains.

**Lemma 3.3.** The positive orthant \( (\mathbb{R}_+)^3 \) is an invariant region for the system (S).

**Proof.** Let us consider \( (x, y, z) : [0, T_0] \rightarrow \mathbb{R}^3 \) a saturated solution for (S) which starts with strictly positive \( x(0), y(0), z(0) \). Under our assumptions (G), it is easy to see that

\[
\begin{aligned}
   x'(t) &\geq x(t) \left[ r - ax(t) - g'_1(0)y(t) \right], \quad 0 \leq t < T_0, \ t \neq (n + l - 1)T, nT; \\
   y'(t) &\geq y(t) \left[ k_1g_1(x(t)) - g'_2(0)z(t) - d_1 \right], \\
   z'(t) &\geq z(t) \left[ k_2g_2(y(t)) - d_2 \right].
\end{aligned}
\]

as long as the solution remains positive. It then follows from Lemma 3.2 that

\[
\begin{aligned}
   x(t) &\geq x(0) \left( 1 - \delta_1 \right) e^{\int_0^t \prod_{0 \leq \tau_k < t} d_k} p(0)ds, \quad 0 \leq t < T_0; \\
   y(t) &\geq y(0) \left( 1 - \delta_2 \right) e^{\int_0^t \prod_{0 \leq \tau_k < t} d_k} p(0)ds, \\
   z(t) &\geq z(0) \left( 1 - \delta_3 \right) e^{\int_0^t \prod_{0 \leq \tau_k < t} d_k} p(0)ds,
\end{aligned}
\]

where \( x(0), y(0), z(0) \) are positive initial data.
where

\[
\begin{aligned}
p_1(t) &= r - ax(t) - g_1'(0)y(t); \\
p_2(t) &= k_1g(x(t)) - g_2'(0)z(t) - d_1; \\
p_3(t) &= k_2g_2(y(t)) - d_2,
\end{aligned}
\]

that is, \(x, y, z\) remain strictly positive on \([0, T_0)\).

Also, using Lemma 3.2, it is possible to show that all solutions of (S) starting in \((\mathbb{R}^*_+)^3\) remain bounded and are actually defined on the whole \(\mathbb{R}^+\).

**Lemma 3.4.** All solutions \((x(\cdot), y(\cdot), z(\cdot))\) of (S) with initial data \((x(\cdot), y(\cdot), z(\cdot))\) in \((\mathbb{R}^*_+)^3\) are bounded and defined on \(\mathbb{R}^+\).

**Proof.** Let us consider a solution \((x(\cdot), y(\cdot), z(\cdot))\) of (S) starting with strictly positive \(x(0), y(0), z(0)\) and define \(u_1: \mathbb{R}^+ \rightarrow \mathbb{R}^+_1\) by

\[u_1(t) = x(t) + \frac{1}{k_1}y(t) + \frac{1}{k_1k_2}z(t), \quad t \geq 0.
\]

One then has

\[
\frac{du_1}{dt} = x(t)[r - ax(t)] - \frac{d_1}{k_1}y(t) - \frac{d_2}{k_1k_2}z(t), \quad t > 0, \quad t \neq (n + l - 1)T, \quad t \neq nT.
\]

Let us denote \(D = \min(d_1, d_2)\). It follows that

\[
\frac{du_1}{dt} + Du_1 \leq x(t) r + D - ax(t), \quad t > 0, \quad t \neq (n + l - 1)T, \quad t \neq nT.
\]

As the right-hand side of (3.5) is bounded from above by \(C = (r + D)^2/(4a)\), it follows that

\[
\frac{du_1}{dt}(t) + Du_1(t) \leq C, \quad t > 0, \quad t \neq (n + l - 1)T, \quad t \neq nT,
\]

together with

\[
u_1((n + l - 1)T^+) \leq (1 - \delta)u_1((n + l - 1)T)
\]

and

\[
u_1(nT^+) = u_1(nT) + \frac{\mu}{k_1k_2},
\]

where \(\delta = \min(\delta_1, \delta_2, \delta_3)\). By Lemma 3.2, it follows that

\[
u_1(t) \leq u_1(0^+) \left[ \prod_{0 < (n + l - 1)T < t} (1 - \delta) \right] e^{-Dt}
\]

\[+ C \int_0^t \left[ \prod_{s \leq (n + l - 1)T < t} (1 - \delta) \right] e^{-D(t-s)} ds + \sum_{0 < nT < t} \frac{\mu}{k_1k_2} e^{-D(t-nT)}, \quad t > 0,
\]

which yields

\[
u_1(t) \leq u_1(0^+) e^{-Dt} + \frac{C(1 - e^{-Dt})}{D} + \frac{\mu}{k_1k_2} e^{DT} - 1, \quad t > 0,
\]
and since the limit of the right-hand side of (3.7) for $t \to \infty$ is

$$L = \frac{C}{D} + \frac{\mu}{k_1 k_2} \frac{e^{DT}}{e^{DT} - 1} < \infty,$$

it easily follows that $u$ is bounded on its domain. Consequently, $x, y, z$ are bounded and it follows by an easy continuability argument that they are defined on the whole $\mathbb{R}_+$. 

It is also important to note that by a very similar procedure one may obtain that

$$u_2(t) \leq u_2(0+) e^{-d_1 t} + \frac{C_1}{d_1} (1 - e^{-d_1 t}),$$

where $u_2(t) = x(t) + \frac{1}{k_1} y(t)$ and $C_1 = (r + d_1)^2/(4a)$. Consequently,

$$x(t) + \frac{1}{k_1} y(t) \leq \left[ x(0+) + \frac{1}{k_1} y(0+) \right] + \frac{(r + d_1)^2}{4ad_1} \quad \text{for } t > 0.$$

At this point, it is useful to note that from the above it may be seen that $y$ is bounded, with boundedness constant

$$(3.8) \quad B = k_1 \left[ x(0+) + \frac{1}{k_1} y(0+) + \frac{(r + d_1)^2}{4ad_1} \right].$$

Of course, this boundedness constant is not necessarily optimal, but it is important to note that it is $T$ and $\mu$-independent. In the following, we shall also be interested in finding an ultimate boundedness constant for $y$, that is, a boundedness constant for $y$ after the transient effects of the initial data are eliminated, rather than a boundedness constant; the former can be much smaller.

We now introduce a few basic results regarding the Floquet theory of impulsive systems of ordinary differential equations. Let us consider the system

$$(3.9) \quad \begin{cases} X'(t) = A(t)X(t), & t \neq \tau_k, \ t \in \mathbb{R}; \\ \Delta X = B_k X, & t = \tau_k, \ \tau_k < \tau_{k+1}, \ k \in \mathbb{Z}. \end{cases}$$

under the following hypotheses.

(H1) $A(\cdot) \in PC(\mathbb{R}, M_n(\mathbb{R}))$ and there is $T > 0$ such that $A(t + T) = A(t)$ for all $t \geq 0$.

(H2) $B_k \in M_n(\mathbb{R})$, det$(I_n + B_k) \neq 0$ for $k \in \mathbb{Z}$.

(H3) There is $q \in \mathbb{N}$ such that $B_{k+q} = B_k$, $\tau_{k+q} = \tau_k + T$ for $k \in \mathbb{Z}$.

Let $\Phi(t)$ be a fundamental matrix of (3.9). Then there is a unique nonsingular matrix $M \in M_n(\mathbb{R})$ such that $\Phi(t + T) = \Phi(t)M$ for all $t \in \mathbb{R}$, which is called the monodromy matrix of (3.9) corresponding to $\Phi$. Actually, all monodromy matrices of (3.9) are similar and consequently they have the same eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, which are called the Floquet multipliers of (3.9). Under these hypotheses, the following result holds.

**Lemma 3.5.** (2) Suppose that conditions (H1)-(H3) hold. Then

1. The system (3.9) is stable if and only if all Floquet multipliers $\lambda_k$, $1 \leq k \leq n$ satisfy $|\lambda_k| \leq 1$ and if $|\lambda_k| = 1$, then to $\lambda_k$ there corresponds a simple elementary divisor.

2. The system (3.9) is asymptotically stable if and only if all Floquet multipliers $\lambda_k$, $1 \leq k \leq n$ satisfy $|\lambda_k| < 1$. 
3. The system (3.9) is unstable if there is a Floquet multiplier $\lambda_k$ such that $|\lambda_k| > 1$.

Note that we shall be able to use Lemma 3.5 in our settings even though we also employ the constant impulse $\Delta z = \mu$ apparently not covered by our Lemma 3.5. This happens since we shall actually study the stability of certain periodic discontinuous solutions by means of the small amplitude perturbations method and the above-mentioned jump condition disappears after we reduce our problem to the stability of the null solution for certain systems, using the proper change of variables which involves discontinuous functions.

4. Periodically forced subsystems. When the intermediate consumer $y$ is eradicated, it is easy to see that the equations in (S) decouple and we are led to consider the properties of the subsystems

$$\begin{align*}
(RS;x) &= \begin{cases} 
  x'(t) = x(t)[r - ax(t)], & t \neq nT, (n + l - 1)T; \\
  \Delta x(t) = -\delta_1 x(t), & t = (n + l - 1)T; \\
  \Delta x(t) = 0, & t = nT; \\
  x(0+) = x_0,
\end{cases} \\
(RS;z) &= \begin{cases} 
  z'(t) = -dz(t), & t \neq nT, (n + l - 1)T; \\
  \Delta z(t) = -\delta_3 z(t), & t = (n + l - 1)T; \\
  \Delta z(t) = \mu, & t = nT; \\
  z(0+) = z_0,
\end{cases}
\end{align*}$$

which describe the dynamics of the resource and of the top predator, respectively, in the absence of the intermediate consumer. First, it is seen that if a certain inequality which characterizes resource productivity is satisfied, then the system formed with the first three equations in (RS;x) has a periodic solution, to which all solutions of (RS;x) starting with strictly positive $x_0$ tend as $t \to \infty$, while if the opposite inequality is satisfied, then all solutions of (RS;x) tend to 0 as $t \to \infty$. The above-mentioned periodic solution will be labeled as $x^*_r$ rather than, for instance, $x^*_{r_a,\delta_1}$, as systems of type (RS;x) will always occur with the same $a$ and $\delta_1$, but sometimes with different $r$’s, so there is no danger of confusion.

**Lemma 4.1.** The following statements hold.

1. Suppose that $\text{ln}(1 - \delta_1) + rT > 0$. Then the system formed with the first three equations in (RS;x) has a periodic solution $x^*_r$. With this notation, the following properties are satisfied.
   (a) $\int_0^T x^*_r(t)dt = (1/a) \text{ln}(1 - \delta_1) + rT$.
   (b) $\lim_{t \to \infty} |x(t) - x^*_r(t)| = 0$ for all solutions $x(t)$ of (RS;x) starting with strictly positive $x_0$.
   (c) $\sup_{t \geq 0} |x^*_r(t)| \leq f_1(r_1, r_2; T, a, \delta_1)$, with $\lim_{r_1 \to r_2} f_1(r_1, r_2; T, a, \delta_1) = 0$.

2. Suppose that $\text{ln}(1 - \delta_1) + rT \leq 0$. Then $\lim_{t \to \infty} x(t) = 0$ for all solutions of (RS;x).

**Proof.** First, it is easy to see that

$$u(t) = \frac{(r/a)u(t_0)e^{r(t-t_0)}}{(r/a) + u(t_0)(e^{r(t-t_0)} - 1)}, \quad (n + l - 1)T < t_0 \leq t \leq (n + l)T$$

Note that we shall be able to use Lemma 3.5 in our settings even though we also employ the constant impulse $\Delta z = \mu$ apparently not covered by our Lemma 3.5. This happens since we shall actually study the stability of certain periodic discontinuous solutions by means of the small amplitude perturbations method and the above-mentioned jump condition disappears after we reduce our problem to the stability of the null solution for certain systems, using the proper change of variables which involves discontinuous functions.
for any solution \( u \) of the first equation in (RS; \( x \)), and so
\[
\frac{(r/a)u((n + l - 1)T +)e^{rT}}{r/a + u((n + l - 1)T +) (e^{rT} - 1)}.
\]
Then
\[
\frac{(r/a)u((n + l - 1)T +)e^{rT}}{r/a + u((n + l - 1)T +) (e^{rT} - 1)}
\]
for any solution \( u \) of the first equation in (RS; \( x \)). Suppose now that \( \ln(1 - \delta_1) + rT > 0 \).

By the periodicity requirement, it follows that
\[
x^*_r((n + l)T +) = \frac{(r/a) \left[ (1 - \delta_1)e^{rT} - 1 \right]}{e^{rT} - 1}.
\]
and so
\[
x^*_r(lT +) = \frac{(r/a) \left[ (1 - \delta_1)e^{rT} - 1 \right]}{e^{rT} - 1}.
\]
Obviously, as \( \ln(1 - \delta_1) + rT > 0 \), the periodic solution \( x^*_r \) does indeed exist, is unique and strictly positive. Actually, it may be seen that
\[
x^*_r(t) = \frac{(r/a)Ae^{(t-(n+l-1)T)}}{1 + A \left( e^{(t-(n+l-1)T)} - 1 \right)}, \quad (n + l - 1)T < t \leq (n + l)T,
\]
where
\[
A = \frac{(1 - \delta_1)e^{rT} - 1}{e^{rT} - 1}.
\]
Also, since
\[
\frac{x^*_r(l)}{x^*_r(t)} = r - ax^*_r(t), \quad t \in ((n + l - 1)T, (n + l)T],
\]
it follows that
\[
\ln \left( x^*_r(t_2) \right) - \ln \left( x^*_r(t_1) \right) = \int_{t_1}^{t_2} [r - ax^*_r(s)] ds; \quad (n + l - 1)T < t_1 \leq t_2 \leq (n + l)T
\]
and so
\[
\ln \left( x^*_r((n + l)T) \right) - \ln \left( x^*_r((n + l - 1)T +) \right) = \int_{(n+l-1)T}^{(n+l)T} [r - ax^*_r(s)] ds.
\]
By the periodicity of \( x^*_r \), it follows that
\[
- \ln (1 - \delta_1) = \int_{0}^{T} [r - ax^*_r(s)] ds,
\]
from which the first assertion follows. Note also that
\[
\ln (1 - \delta_1) + \int_{0}^{T} [r - 2ax^*_r(s)] ds = - (\ln (1 - \delta_1) + rT) < 0.
\]
Let now $x(t)$ be a solution of (RS; $x$) starting with strictly positive initial data. We shall prove that $\lim_{t \to \infty} |x(t) - x^*_n(t)| = 0$.

If $x(lT^+) = x^*_n(lT^+)$, then obviously $x \equiv x^*_n$. Suppose now that $x(lT^+) > x^*_n(lT^+)$; if the reverse inequality is satisfied one can devise a similar argument to obtain the conclusion mentioned above.

Let us denote $f : \mathbb{R}_+ \to \mathbb{R}_+$,

$$f(x) = (1 - \delta_1) \frac{(r/a)x^T}{(r/a) + x(e^r - 1)}.$$  

It is then seen that $x \mapsto f(x)$ is strictly increasing on $\mathbb{R}_+$, while $x \mapsto f(x)/x$ is strictly decreasing on $\mathbb{R}_+$. By (4.1), it is also seen that

$$x^*_n((l+1)T^+) = f(x^*_n(lT^+)),$$

and by the periodicity of $x^*_n$ it is seen that $x^*_n((l+1)T^+) = x^*_n(lT^+)$. It follows that

$$x((l+1)T^+) = f(x(lT^+)) > f(x^*_n(lT^+)) = x^*_n(lT^+),$$

since $f$ is strictly increasing on $\mathbb{R}_+$. Also,

$$x((l+1)T^+) = f(x(lT^+)) = \frac{f(x(lT^+))}{x(lT^+)} x(lT^+) < x(lT^+),$$

since $x(lT^+) > x^*_n(lT^+)$, $f(x^*_n(lT^+)) = x^*_n(lT^+)$ and $x \mapsto f(x)/x$ is strictly decreasing on $\mathbb{R}_+$.

Similarly, by an induction argument,

$$x((n+1)T^+) = f(x((n+1)T^+)) > f(x^*_n((n+1)T^+)) = f(x^*_n(lT^+)) = x^*_n(lT^+)$$

and

$$x((n+1)T^+) = f(x((n+1)T^+)) = \frac{f(x((n+1)T^+))}{x((n+1)T^+)} x((n+1)T^+) < x((n+1)T^+).$$

One then obtains that $(x((n+l)T^+))_{n \geq 0}$ is monotonically decreasing and bounded from below by $x^*_n(lT^+)$, so it is convergent to some $w_1 > 0$. Also,

$$x((n+l+1)T^+) - x((n+l)T^+) = f(x((n+l)T^+)) - x((n+l)T^+) \to 0 \quad \text{as} \quad n \to \infty.$$  

From the above, it follows that $f(w_1) = w_1$, and so $w_1 = x^*_n(lT^+)$. Also, by (4.1), one may prove that

$$[x(t) - x^*_n(t)] \leq e^{rT} [x((n+l)T^+) - x^*_n((n+l)T^+)], \quad \text{for} \quad t \in ((n+l)T^+, (n+l+1)T^+]$$

from which the second assertion follows. The remaining assertion can be proved by direct computation.
Suppose now that \( \ln(1 - \delta_1) + rT \leq 0 \). Again, by (4.2), it is seen that

\[
x((n + l)T^+) = (1 - \delta_1)\frac{(r/a)x((n + l - 1)T^+e^{rT})}{(r/a) + x((n + l - 1)T^+e^{rT} - 1)}
\]

\[
= x((n + l - 1)T^+)\frac{(r/a)(1 - \delta_1)e^{rT}}{(r/a) + x((n + l - 1)T^+e^{rT} - 1)}
\]

\[
\leq x((n + l - 1)T^+),
\]

as \((1 - \delta_1)e^{rT} \leq 1\). It then follows that \((x((n + l - 1)T^+))_{n \geq 0}\) is monotonically decreasing and bounded from below by 0, so it converges to some \(w_2 \geq 0\). Since \(x((n + l)T^+) = f(x((n + l - 1)T^+))\), it follows that \(f(w_2) = w_2\). Then

\[
w_2 = w_2\frac{(r/a)(1 - \delta_1)e^{rT}}{(r/a) + w_2(e^{rT} - 1)}
\]

and, since \(\ln(1 - \delta_1) + rT \leq 0\), it easily follows that \(w_2 = 0\). By (4.1), it also follows that

\[
x(t) \leq x((n + l - 1)T^+)e^{rT} \quad \text{for} \quad t \in ((n + l - 1)T, (n + l)T]
\]

and so \(\lim_{t \to \infty} x(t) = 0\). \(\square\)

We now suggest an approximate interpretation of the hypotheses in Lemma 4.1. Let us suppose that \(x\) approaches 0 in \((RS;x)\). Then \(rT\) approximates the total growth (per unit biomass) of the resource biomass in a period, while \(\ln(1 - \delta_1)\) is a correction term which accounts for the loss of resource biomass (per unit biomass) due to pesticide spraying. If the total growth \(rT\) does not exceed the loss \(\ln(1 - \delta_1)\), there is a net loss of resource biomass when \(x\) approaches 0 and so the resource biomass \(x(t)\) tends to 0 as \(t \to \infty\), while if \(\ln(1 - \delta_1) + rT > 0\), there is a net gain of resource biomass when \(x\) approaches 0 which prevents the extinction of the resource \(x\).

Secondly, it is seen that the system formed with the first three equations in \((RS;z)\) has a periodic solution to which all solutions of \((RS;z)\) starting with strictly positive \(z_0\) tend as \(t \to \infty\), irrespective of the sign of \(\ln(1 - \delta_1) + rT\). This happens since the survival of the top predator is assured by the periodic impulse \(\mu\) and does not depend upon the survival or extinction of the resource, although the persistence level is, of course, indirectly affected. Again, this solution will be labeled as \(z^*_{d_2}\), for reasons similar to those outlined above.

**Lemma 4.2.** The system formed with the first three equations in \((RS;z)\) has a periodic solution \(z^*_{d_2}\). With this notation, the following properties are satisfied.

1. \(\int_0^T z^*_{d_2}(t)dt = \frac{\mu}{1 - e^{-d_2T}} \left[ (1 - e^{-d_2T}) + (1 - \delta_3)(e^{-d_2T} - e^{-d_2T}) \right]\).

2. \(\lim_{t \to \infty} \left| z(t) - z^*_{d_2}(t) \right| = 0\) for all solutions \(z(t)\) of \((RS;z)\) starting with strictly positive \(z_0\).

3. \(\sup_{t \geq 0} \left| z^*_{d_2}(t) - z^*_{d_2}(t) \right| \leq f_2(d_2, \tilde{d}_2; T, a, \delta_3), \quad \text{with} \quad \lim_{d_2 \to d_2} f_2(d_2, \tilde{d}_2; T, a, \delta_3) = 0\).

**Proof.** First, it is easy to see that

\[
u(t) = e^{-d_2(t - t_0)}u(t_0) \quad t, \ t_0 \in ((n + l - 1)T, nT) \text{ or } (nT, (n + l + 1)T)
\]
for any solution \( u \) of the first equation in (RS; \( z \)) and so

\[
z_{d_2}^*((n + 1)T+) = z_{d_2}^*((n + 1)T) + \mu \\
= e^{-d_2(1-l)T}z_{d_2}^*((n + l)T) + \mu \\
= e^{-d_2(1-l)T}(1 - \delta_3)z_{d_2}^*((n + l)T) + \mu \\
= e^{-d_2(1-l)T}(1 - \delta_3)e^{-d_2T}z_{d_2}^*(nT+) + \mu \\
= e^{-d_2T}(1 - \delta_3)z_{d_2}^*(nT+) + \mu.
\]

By the periodicity requirement, it follows that

\[
z_{d_2}^*(nT+) = e^{-d_2T}(1 - \delta_3)z_{d_2}^*(nT+) + \mu
\]

and so

\[
(4.6) \\
z_{d_2}^*(0+) = \frac{\mu}{1 - e^{-d_2T}(1 - \delta_3)}.
\]

Obviously, by (4.6), the periodic solution searched for does indeed exist, is unique and strictly positive. Actually, it may be seen that

\[
z_{d_2}^* = \begin{cases} \\
\frac{\mu}{1 - e^{-d_2T}(1 - \delta_3)}e^{-d_2(t-nT)}, & t \in (nT, (n + l)T] \\
e^{-d_2T}(1 - \delta_3), & t \in ((n + l)T, (n + 1)T].
\end{cases}
\]

The first assertion follows then by direct computation.

Let now \( z \) be a solution of (RS; \( z \)) with strictly positive initial data. We shall prove that \( \lim_{t \to \infty} |z(t) - z_{d_2}^*(t)| = 0 \).

It is seen that \( z - z_{d_2}^* \) verifies the system

\[
\begin{cases} \\
(z - z_{d_2}^*)(t) = -d_2(z - z_{d_2}^*)'(t), & t \neq (n + l - 1)T, t \neq nT; \\
\Delta(z - z_{d_2}^*)(t) = -\delta_3(z - z_{d_2}^*)'(t), & t = (n + l - 1)T; \\
\Delta(z - z_{d_2}^*)(t) = 0, & t = nT.
\end{cases}
\]

Consequently,

\[
(4.8) \\
z(t) - z_{d_2}^*(t) = \begin{cases} \\
\frac{e^{-d_2(t-(n-1)T)}}{1 - e^{-d_2T}(1 - \delta_3)}(z(0+) - \frac{\mu}{1 - e^{-d_2T}(1 - \delta_3)}) (1 - \delta_3)^{n-1}, & t \in ((n-1)T, (n + l)T]; \\
e^{-d_2(t-(n-1)T)}(z(0+) - \frac{\mu}{1 - e^{-d_2T}(1 - \delta_3)}) (1 - \delta_3)^n, & t \in ((n + l - 1)T, nT];
\end{cases}
\]

from which the second assertion follows. The third assertion can be proved by direct computation.

5. Local stability results. A Floquet analysis. In this section we study the local stability of the resource and intermediate consumer-free periodic solution \((0, 0, z_{d_2}^*(t))\) and of the intermediate consumer-free periodic solution \((x_{d_2}^*(t), 0, z_{d_2}^*(t))\) by means of Floquet theory, supposing that the productivity condition for the resource \(\ln(1 - \delta_1) + rT > 0\) is satisfied. In this sense, it will be seen that the local stability of the intermediate consumer-free periodic solution is governed by a threshold-like condition expressed in terms of an integral involving the periodic solutions \(x_{d_2}^*\) and \(z_{d_2}^*\) introduced in the previous section, while the resource and intermediate consumer-free periodic solution is always unstable.

**Theorem 5.1.** Suppose that \(\ln(1 - \delta_1) + rT > 0\). The following properties hold.
1. The resource and intermediate consumer-free periodic solution \((0, 0, z^*_{d_2}(t))\) is unstable.
2. The intermediate consumer-free periodic solution \((x^*_1(t), 0, z^*_{d_2}(t))\) is locally asymptotically stable provided that

\[
\ln(1 - \delta_2) + \int_0^T [k_1g_1(x^*_1(s)) - g'_2(0)z^*_w(s) - d_1] \, ds < 0
\]

and unstable provided that the reverse inequality holds.

**Proof.** To study the stability of the resource and intermediate consumer-free periodic solution \((0, 0, z^*_{d_2}(t))\), let us denote

\[
x(t) = u(t), \quad y(t) = v(t), \quad z(t) = w(t) + z^*_{d_2}(t),
\]

\(u, v\) and \(w\) being understood as small amplitude perturbations. Substituting (5.2) into the first three equations of (S), one obtains

\[
\begin{align*}
    \dot{w}(t) &= u(t)[r - au(t)] - g_1(u(t))v(t) \\
    \dot{v}(t) &= k_1g_1(u(t)) - g_2(v(t)) [w(t) + z^*_{d_2}(t)] - d_1v(t) \\
    \dot{u}(t) &= k_2g_2(v(t)) [w(t) + z^*_{d_2}(t)] - d_2w(t)
\end{align*}
\]

The corresponding linearization of (5.3) at \((0, 0, 0)\) is

\[
\begin{align*}
    \dot{u}(t) &= ru(t) \\
    \dot{v}(t) &= -[g'_2(0)z^*_{d_2}(t) + d_1] v(t) \\
    \dot{w}(t) &= k_2g_2(0)z^*_{d_2}(t)v(t) - d_2w(t)
\end{align*}
\]

and so a fundamental matrix of (5.4) is

\[
\Phi^1_{1}(t) = \begin{pmatrix}
    e^{rt} & e^{-\int_0^t[u'(s)z^*_{d_2}(s) + d_1] \, ds} & 0 \\
    0 & e^{-\int_0^t[g'_2(0)z^*_w(s) + d_1] \, ds} & 0 \\
    0 & 0 & e^{-d_2t}
\end{pmatrix}.
\]

The linearization of the jump conditions at \((n + l - 1)T\) reads as

\[
\begin{align*}
    \Delta u(t) &= -\delta_1u(t), \quad t = (n + l - 1)T; \\
    \Delta v(t) &= -\delta_2v(t), \\
    \Delta w(t) &= -\delta_3w(t),
\end{align*}
\]

while the linearization of the jump conditions at \(nT\) reads as

\[
\begin{align*}
    \Delta u(t) &= 0, \quad t = nT; \\
    \Delta v(t) &= 0, \\
    \Delta w(t) &= 0.
\end{align*}
\]

Consequently, the local stability of the resource and intermediate consumer-free periodic solution \((0, 0, z^*_{d_2}(t))\) can be analyzed by studying the eigenvalues of the monodromy matrix

\[
M_1 = \begin{pmatrix}
    1 - \delta_1 & 0 & 0 \\
    0 & 1 - \delta_2 & 0 \\
    0 & 0 & 1 - \delta_3
\end{pmatrix} \Phi^1_{1}(T).
\]
Since the eigenvalues of $M_1$ are
\[
\lambda_1 = (1 - \delta_1)e^{\varphi T}, \quad \lambda_2 = (1 - \delta_2)e^{-\int_0^t [g_2^r(0)z_{d_2}^*(s) + d_1]ds}, \quad \lambda_3 = (1 - \delta_3)e^{-d_2 T}
\]
and $\lambda_1 > 1$, it follows that the resource and intermediate consumer-free periodic solution $(0, 0, z_{d_2}^*(t))$ is unstable, with an one-dimensional unstable manifold.

We now study the stability of the intermediate consumer-free periodic solution $(x_{\ast}^*(t), 0, z_{d_2}^*(t))$. Let us denote
\[
(5.8) \quad x(t) = u(t) + x_{\ast}^*(t), \quad y(t) = v(t), \quad z(t) = w(t) + z_{d_2}^*(t),
\]
u, v, w being understood again as small amplitude perturbations. Substituting (5.8) into the first three equations of (S), one obtains
\[
\begin{align*}
\Phi^r_{L}(t) &= \begin{pmatrix}
\left(\int_0^t e^{\varphi(s)}g_1(x_{\ast}^*(s))e^{\psi(s)-\varphi(s)}ds\right) & 0 \\
0 & e^{\psi(t)} & 0 \\
0 & e^{-d_2 T} & 0
\end{pmatrix}
\end{align*}
\]

The linearization of the jump conditions at $(n + l - 1)T$ and $(n + l)T$ gives again (5.6) and (5.7). Consequently, the local stability of the intermediate consumer-free periodic solution $(x_{\ast}^*(t), 0, z_{d_2}^*(t))$ can be analyzed by studying the eigenvalues of the monodromy matrix
\[
M_2 = \begin{pmatrix}
1 - \delta_1 & 0 & 0 \\
0 & 1 - \delta_2 & 0 \\
0 & 0 & 1 - \delta_3
\end{pmatrix}
\]

It is seen that the eigenvalues of $M_2$ are
\[
\lambda_1 = (1 - \delta_1)e^{\varphi(T)}, \quad \lambda_2 = (1 - \delta_2)e^{\psi(T)}, \quad \lambda_3 = (1 - \delta_3)e^{-d_2 T}.
\]
It is obvious that $0 < \lambda_3 < 1$. Also, $0 < \lambda_1 < 1$, from Lemma 4.1. Consequently, if (5.1) is satisfied, then $0 < \lambda_2 < 1$ and $(x_{\ast}^*(t), 0, z_{d_2}^*(t))$ is stable, while if the reverse of
(5.1) is satisfied, then \( \lambda_2 > 1 \) and \((x_+^*(t), 0, z_{d_2}^*(t))\) is unstable, with an one-dimensional unstable manifold. \( \square \)

Note that the meaning of condition (5.1) is completely similar to that of condition \( \ln(1 - \delta_1) + rt < 0 \), but applied to the dynamics of \( y \) this time. Namely, suppose that \( y \) approaches 0. Then \( \int_0^T \left[ k_1 g_1(x_+^*(s)) - g_2(0) z_{d_2}^*(s) - d_1 \right] ds \) approximates the total growth (per unit biomass) of the intermediate consumer biomass in a period (note that \( \lim_{t \to 0} (g(t)/t) = g_2'(0) \)), while \( \ln(1 - \delta_2) \) is a correction term which accounts for the loss of intermediate consumer biomass (per unit biomass) due to pesticide spraying. If the total growth exceeds the loss \( \ln(1 - \delta_2) \), then there is a net gain of consumer biomass when \( y \) approaches 0 which prevents the extinction of the intermediate consumer, while if the loss \( \ln(1 - \delta_2) \) exceeds the total growth, there is a net loss of consumer biomass when \( y \) approaches 0 and so \( y(t) \) tends to 0 as \( t \to \infty \). Also, condition \( \ln(1 - \delta_1) + rt > 0 \) ensures the instability of the resource and intermediate consumer-free periodic solution, since it prevents the extinction of the resource.

Since \( g_1 \) and \( g_2 \) are general functional responses, we have to state our stability condition (5.1) in terms of the periodic solutions \( x_+^* \) and \( z_{d_2}^* \) rather than in a more explicit form. Actually, this form may make more sense even when the particular forms of \( g_1 \) and \( g_2 \) are known (for instance, when \( g_1, g_2 \) are Holling type II functional responses), as the resulting explicit inequalities look rather cumbersome and their interpretations are not transparent.

6. Global stability results. In this section, we perform a global stability analysis of the resource and intermediate consumer-free periodic solution \((0, 0, z_{d_2}^*(t))\) and of the intermediate consumer-free periodic solution \((x_+^*(t), 0, z_{d_2}^*(t))\), respectively.

**Theorem 6.1.** The following statements hold.

1. Suppose that \( \ln(1 - \delta_1) + rt \leq 0 \). Then the resource and intermediate consumer-free periodic solution \((0, 0, z_{d_2}^*(t))\) is globally asymptotically stable.
2. Suppose that \( \ln(1 - \delta_1) + rt > 0 \). Then the intermediate consumer-free periodic solution \((x_+^*(t), 0, z_{d_2}^*(t))\) is globally asymptotically stable provided that

\[
(6.1) \quad \ln(1 - \delta_2) + \int_0^T \left[ k_1 g_1(x_+^*(s)) - c_{g_2} z_{d_2}^*(s) - d_1 \right] ds < 0,
\]

where

\[
c_{g_2} = \inf_{0 \leq u \leq M_y} g_2'(u),
\]

\( M_y \) being an ultimate boundedness constant for \( y \).

**Proof.** Suppose first that \( \ln(1 - \delta_1) + rt \leq 0 \). Let \( \varepsilon_1 > 0 \) such that \( k_1 g_1(\varepsilon_1) < d_1 \) (this is always possible since \( \lim_{\varepsilon \to 0} g_1(\varepsilon) = 0 \)) and let also \( \eta = (1 - \delta_1) e^{(k_1 g_1(\varepsilon_1) - d_1)T} \).

Note that \( 0 < \eta < 1 \). It is seen that

\[
x'(t) = x(t) [r - ax(t)] - g_1(x(t)) y(t) \leq x(t) [r - ax(t)]
\]

and so, by Lemma 3.1, \( x(t) \leq \tilde{x}(t) \) for \( t \geq 0 \), where \( \tilde{x} \) is the solution of \((\text{RS}; x)\) with the same initial data at 0+ as \( x \). As any such solution \( \tilde{x} \) tends to 0 for \( t \to \infty \), by Lemma 4.1, \( x \) tends to 0 as well and there is \( T_1 > 0 \) such that \( x(t) \leq \varepsilon_1 \) for \( t \geq T_1 \). For the sake of simplicity, we suppose that \( x(t) \leq \varepsilon_1 \) for all \( t > 0 \). One then obtains that

\[
y'(t) = k_1 g_1(x(t)) y(t) - g_2(y(t)) z(t) - d_1 y(t)
\]

\[
\leq y(t) [k_1 g_1(\varepsilon_1) - d_1], \quad t \neq (n + l - 1)T.
\]
By integrating the above inequality on \(((n + l - 1)T, (n + l)T]\), one obtains
\[
\ln (y(n + l)T) - \ln (y(n + l - 1)T^+) \leq (k_1 g_1(\varepsilon_1) - d_1)T \quad \text{for } n \geq 1
\]
and so
\[
\ln (y(n + l)T) - \ln (y(n + l - 1)T) - \ln (1 - \delta_1) \leq (k_1 g_1(\varepsilon_1) - d_1)T \quad \text{for } n \geq 1.
\]
It then follows that
\[
y((n + l)T) \leq y((n + l - 1)T)\eta
\]
and consequently
\[
y((n + l)T) \leq y(IT)\eta^n,
\]
which implies that \(y((n + l)T) \to 0\) as \(n \to \infty\). Also,
\[
y(t) \leq y((n + l - 1)T^+)e^{(k_1 g_1(\varepsilon_1) - d_1)(t - (n + l - 1)T)}, \quad t \in ((n + l - 1)T, (n + l)T]
\]
which implies that
\[
y(t) \leq y((n + l - 1)T^+), \quad t \in ((n + l - 1)T, (n + l)T]
\]
and consequently \(y(t) \to 0\) as \(t \to \infty\).

We finish by proving that \(z(t) - z_{d_2}^*(t) \to 0\) as \(t \to \infty\). To this purpose, let \(0 < \varepsilon_2 < d_2/(k_2 L_2)\). Since \(y(t) \to 0\) as \(t \to \infty\), there is some \(T_2 > 0\) such that \(y(t) \leq \varepsilon_2\) for all \(t \geq T_2\). For the sake of simplicity, we suppose that \(y(t) \leq \varepsilon_2\) for all \(t > 0\). It follows that
\[
z'(t) = k_2 g_2(y(t))z(t) - d_2 z(t)
\leq k_2 L_2 y(t)z(t) - d_2 z(t)
\leq -(d_2 - k_2 L_2 \varepsilon_2)z(t), \quad t \neq (n + l - 1)T, t \neq nT.
\]
Consequently, one infers from Lemma 3.1 that
\[
\tilde{z}_1(t) \leq z(t) \leq \tilde{z}_2(t)
\]
where \(\tilde{z}_1\) is the solution of (RS; \(z\)) with the same initial data at 0+ as \(z\) and \(\tilde{z}_2\) is the solution of (RS; \(z\)) with \(d_2\) changed into \(d_2 - k_2 L_2 \varepsilon_2\) and the same initial data at 0+ as \(z\).

As these solutions become close to \(z_{d_2}^*(t)\), respectively to \(z_{d_2 - k_2 L_2 \varepsilon_2}^*(t)\) as \(t \to \infty\), by Lemma 4.2, it follows that, for \(t\) large enough,
\[
z_{d_2}^*(t) - \varepsilon_2 \leq z(t) \leq z_{d_2 - k_2 L_2 \varepsilon_2}^*(t) + \varepsilon_2
\]
and the conclusion follows from Lemma 4.2. The first assertion is now established.

Suppose now that \(\ln(1 - \delta_1) + rT > 0\). We first show that \(y(t) \to 0\) as \(t \to \infty\). To this purpose, choose \(\varepsilon_3 > 0\) such that
\[
\ln(1 - \delta_2) + \int_0^T \left[ k_1 g_1(x_1^*(s) + \varepsilon_3) - c_{g_2}(z_{d_2}^*(s) - \varepsilon_3) - d_1 \right] ds < 0
\]
For the sake of simplicity, we suppose that $t$ is satisfied. Let us also denote by periodicity. It then follows that

$$\int_0^\infty \Phi(t) ds$$

and observe that $0 < \xi < 1$.

It is seen that

$$x'(t) = x(t) [r - ax(t)] - g_1(x(t))y(t) \leq x(t) [r - ax(t)],$$

and so, by Lemma 3.1, $x(t) \leq \bar{x}(t)$ for $t \geq 0$, where $\bar{x}$ is the solution of (RS;x) with the same initial data at 0+ as $x$. As any such solution becomes close to $x^*_t$ for $t \to \infty$, by Lemma 4.1, there is some $T_3 > 0$ such that $x(t) \leq x^*_t(t) + \varepsilon_3$ for $t \geq T_3$.

For the sake of simplicity, we suppose that $x(t) \leq x^*_t(t) + \varepsilon_3$ for all $t > 0$.

Also,

$$z'(t) = k_2g_2(y(t))z(t) - d_2z(t) \geq -d_2z(t),$$

and so, by Lemma 3.1, $z(t) \geq \bar{z}(t)$ for $t \geq 0$, where $\bar{z}$ is the solution of (RS;z) with the same initial data at 0+ as $z$. As any such solution becomes close to $z^*_t$ for $t \to \infty$, by Lemma 4.2, there is some $T_4 > 0$ such that $z(t) \geq z^*_t(t) - \varepsilon_3$ for $t \geq T_4$.

For the sake of simplicity, we suppose that $z(t) \geq z^*_t(t) - \varepsilon_3$ for all $t > 0$.

Since $y(t)$ is ultimately bounded, there is $T_5 > 0$ such that $y(t) \leq M_y$ for all $t \geq T_5$, where $M_y$ is an ultimate boundedness constant for $y$. For the sake of simplicity, we suppose that $y(t) \leq M_y$ for all $t > 0$. Also, note that in this situation $g_2(y(t)) \geq c_{g_2}y(t)$ for $t \geq 0$. One then obtains that

$$y'(t) = k_1g_1(x(t))y(t) - g_2(y(t))z(t) - d_1y(t)$$

$$\leq y(t) \left[ k_1g_1(x^*_t(t) + \varepsilon_3) - c_{g_2}(z^*_t(t) - \varepsilon_3) - d_1 \right], \quad t \neq (n + l - 1)T,$$

and it consequently follows that

$$\frac{y'(t)}{y(t)} \leq k_1g_1(x^*_t(t) + \varepsilon_3) - c_{g_2}(z^*_t(t) - \varepsilon_3) - d_1, \quad t \neq (n + l - 1)T.$$

By integrating the above inequality on $((n + l - 1)T, (n + l)T]$, one obtains

$$\ln(y((n + l)T)) - \ln(y((n + l - 1)T))$$

$$\leq \int_{(n + l - 1)T}^{(n + l)T} \left[ k_1g_1(x^*_t(t) + \varepsilon_3) - c_{g_2}(z^*_t(t) - \varepsilon_3) - d_1 \right] dt$$

and so

$$\ln(y((n + l)T)) - \ln(y((n + l - 1)T)) - \ln(1 - \delta_2)$$

$$\leq \int_0^T \left[ k_1g_1(x^*_t(t) + \varepsilon_3) - c_{g_2}(z^*_t(t) - \varepsilon_3) - d_1 \right] dt$$

by periodicity. It then follows that

$$y((n + l)T) \leq y((n + l - 1)T) \xi$$

and consequently

$$y((n + l)T) \leq y(IT) \xi^n.$$
which implies that \( y((n + l)T) \to 0 \) as \( n \to \infty \). Also
\[
\frac{y'(t)}{y(t)} \leq k_1g_1(x(t)) \leq \tilde{k},
\]
\( \tilde{k} \) being a suitable boundedness constant, so
\[
y(t) \leq y((n + l - 1)T + )e^{\tilde{k}(t+(n+l-1)T)}, \quad t \in ((n + l - 1)T, (n + l)T]
\]
which implies that
\[
y(t) \leq (1 - \delta_2)y((n + l - 1)T)e^{\tilde{k}T}, \quad t \in ((n + l - 1)T, (n + l)T],
\]
and consequently \( y(t) \to 0 \) as \( t \to \infty \). We now prove that \( x(t) - x^*_n(t) \to 0 \) as \( t \to \infty \).

To this purpose, let \( 0 < \varepsilon_4 \leq r/L_1 \). Since \( y(t) \to 0 \) as \( t \to \infty \), there is \( T_6 > 0 \) such that \( y(t) < \varepsilon_4 \) for \( t \geq T_6 \). For the sake of simplicity, we suppose that \( y(t) < \varepsilon_4 \) for all \( t > 0 \).

It follows that
\[
x'(t) = x(t) [r - ax(t)] - g_1(x(t))y(t)
\]
\[
= x(t) \left[ r - \frac{g_1(x(t))}{x(t)} y(t) - ax(t) \right]
\]
\[
\geq x(t) \left[ (r - L_1\varepsilon_4) - ax(t) \right]
\]
for \( t \neq (n + l - 1)T, t \neq nT \). Consequently, one infers from Lemma 3.1 that
\[
\tilde{x}_1(t) \leq x(t) \leq \tilde{x}_2(t)
\]
where \( \tilde{x}_2 \) is the solution of (RS;\( x \)) with the same initial data at \( 0+ \) as \( x \) and \( \tilde{x}_1 \) is the solution of (RS;\( x \)) with \( r \) changed into \( r - L_1\varepsilon_4 \) and the same initial data at \( 0+ \) as \( x \).

As these solutions become close to \( x^*_n(t) \), respectively to \( x^*_n-L_1\varepsilon_4(t) \) as \( t \to \infty \), by Lemma 4.1, it follows that, for \( t \) large enough,
\[
x^*_n-L_1\varepsilon_4(t) - \varepsilon_4 \leq x(t) \leq x^*_n(t) + \varepsilon_4
\]
and the conclusions now follow again from Lemma 4.1. To prove that \( z(t) - z^*_n(t) \to 0 \) as \( t \to \infty \), we may proceed as done for the proof of the first assertion. The second assertion is now established.

Note that condition (6.1) has a somewhat theoretical value and is only sufficient for the global asymptotic stability of the intermediate consumer-free periodic solution. One may not expect, though, an integral condition of type (6.1) to be threshold-like (to be necessary as well). This happens since (S) has to inherit, at least partially, the chaotic behavior of the unperturbed system, which is attained for a certain window in the parameter space, as noted in Klebanoff and Hastings [11]. At this point, the availability of a good estimate of the ultimate boundedness constant for \( y \) or of \( c_{g_2} \) is crucial. In this regard, if one considers the case in which \( g_2 \) is a Holling type II functional response, \( g_2(y) = (a_2y)/(1 + b_2y) \), then \( g_2'(y) = a_2/(1 + b_2y)^2 \) and then
\[
\inf_{y \in \mathbb{R}_+} g_2'(u) = 0.
\]
Consequently, if no good estimations for the ultimate boundedness constant are available and \( B \) is large, then the only sensible way to ensure the validity of (6.1) is to assume that
\[
\ln(1 - \delta_2) + \int_0^T [k_1g_1(x^*_n(s)) - d_1] ds < 0,
\]
but this is a rather crude estimation, which ensures the extinction of the intermediate consumer even if no top predator is present.

Note also that, at least formally, both the local stability condition (5.1) and the global stability condition (6.1) display a significant dependence on the functional response \( g_2 \) of the top predator, with a dominance on the dependence on \( a_2 \).

7. Biological interpretations of the stability results. From Theorem 6.1, we note that if the pesticide is not selective enough, that is, if \( \delta_1 \) is large enough to make \( \ln(1 - \delta_1) + rT \) negative, or, in other words, if the pesticide has a significant negative effect on the growth of the resource biomass, then the resource and intermediate consumer-free periodic solution is globally asymptotically stable, which means that our control strategy fails. Alternatively, this means that a non-selective pesticide should not be applied very often (\( T \) should be large) in order to avoid resource extinction. Of course, this may have a negative impact on the overall success of the integrated pest management strategy.

From Theorem 6.1, it is seen that, theoretically speaking, our control strategy can be always made to succeed globally by the use of proper pesticides, provided that \( \delta_1 \) is small enough, in order to have the inequality \( \ln(1 - \delta_1) + rT > 0 \) satisfied, and \( \delta_2 \) is large enough to have (6.1) satisfied, for any given top predator functional response \( g_2 \).

Also, it is seen that an aggressive \( (g_2'(0) \) large enough) top predator may stabilize an otherwise unstable intermediate consumer-free periodic solution, at least locally (see (5.1)). In order to stabilize the intermediate consumer-free periodic solution globally, the top predator should be aggressive enough, even at large intermediate consumer densities, when saturation effects are supposed to appear, so that

\[
\inf_{0 \leq u \leq M_y} g_2'(u) > \frac{\ln(1 - \delta_2) + \int_0^T [k_1 g_1(x^*_r(s)) - d_1] ds}{\int_0^T z_2^*(s) ds}.
\]

If \( g_2 \) is a Holling type II functional response (see above) or a Ivlev functional response \( (g_2(x) = k(1 - e^{-bx})) \), which are convex regarded as functions of \( x \), then the above reduces to

\[
g_2'(M_y) > \frac{\ln(1 - \delta_2) + \int_0^T [k_1 g_1(x^*_r(s)) - d_1] ds}{\int_0^T z_2^*(s) ds}.
\]

Note that \( \int_0^T k_1 g_1(x^*_r(s)) ds \) and \( \int_0^T z_2^*(s) ds \) are \( g_2 \)-independent.

Since \( \lim_{\mu \to \infty} \int_0^T z_2^*(s) ds = +\infty \), from Lemma 4.2, and \( x^*_r \) does not depend upon \( \mu \), it is seen from (6.1) that the intermediate consumer-free periodic solution can be stabilized globally by means of increasing \( \mu \) alone. Note that \( B \), the global boundedness constant for \( y \) which is indicated in (3.8) and which may also serve as an ultimate boundedness constant for \( y \), is \( \mu \)-independent.

Also,

\[
\int_0^T k_1 g_1(x^*_r(s)) ds < k_1 L_1 \int_0^T x^*_r(s) ds = k_1 L_1 (1/a) (\ln(1 - \delta_1) + rT),
\]

from Lemma 4.1, so

\[
\limsup_{T \to (\ln(1 - \delta_1))/r} \int_0^T k_1 g_1(x^*_r(s)) ds \leq 0.
\]
As

$$\liminf_{T \to \infty} \frac{1}{T} \int_0^T (z_{d_2}^*(s) + d_1)ds > 0$$

from Lemma 4.2 and $c_{g_2}$ is $T$-independent, it is seen from (6.1) that the intermediate consumer-free periodic solution can also be globally stabilized by means of decreasing $T$ alone, in such a way that $\ln(1 - \delta_1) + rT$ remains strictly positive.

However, as mentioned in the Introduction, our purpose is to drive the intermediate consumer population under the economic injury level rather to eradicate it completely, so our pest management strategy may be considered successful even in situations in which (6.1) is not satisfied, provided that the intermediate consumer population stabilizes under the economic injury level.

Accepting $(1/T) \int_0^T f(t)dt$ as an averaging measure for the oscillations of a periodic and positive function $f$ of period $T$ (an average level of persistence, that is), it is seen from Lemma 4.2 that an increase in $\mu$ causes an increase in the average level of $z_{d_2}^*$, while from Lemma 4.1 is is seen that an increase in $\mu$ has no effect on the average level of $x_1^*$.

From Lemma 4.1, it may also be observed that an increase in the carrying capacity of the environment (a decrease of $a$ while keeping $r$ constant, that is) causes an increase in the average level of $x_1^*$, while having no effect on the average level of $z_{d_2}^*$. This is certainly conceivable, since if $y$ tends to extinction, then the resource $x$ and the top predator $z$ are essentially independent, as the top predator $z$ does not feed upon the resource $x$. Also, as seen from (5.1) and (6.1), an increase in the carrying capacity of the environment may not necessarily destabilize the intermediate consumer-free periodic solution $(x_1^*(t), 0, z_{d_2}^*(t))$, at least when the functional response $g_1$ of the intermediate consumer is a Holling type II functional, since $\int_0^T g_1(x_1^*(s))ds$ is bounded from above as a function of $a$, but it certainly reduces the chances of having a stable intermediate consumer-free periodic solution, since $\int_0^T g_1(x_1^*(s))ds$ is decreasing as a function of $a$. It is then seen that we obtain a paradox of enrichment for our food chain model, albeit in a weaker form. Also, noting that all terms in (5.1) are negative except for $k_1 \int_0^T g_1(x_1^*(s))ds$, we observe that periodic solutions $(x_1^*(t), 0, z_{d_2}^*(t))$ with low $x_1^*$'s are inherently stable rather than unstable, so the paradox of biological control is not present in our model.

To show that our pest management strategy does not overly rely on the use of pesticides, although this, in some sense, has already been observed above, we briefly study below the case in which no pesticides are sprayed (that is, $\delta_1 = \delta_2 = \delta_3 = 0$) and outline the success conditions.

It is seen that in this situation $\ln(1 - \delta_1) + rT = rT > 0$ and $\int_0^T z_{d_2}^*(s)ds = \mu/d_2$. Also, this time $x_1^*(t) = r/a$ for $t \geq 0$ (see (4.3) and (4.4)). We consequently obtain with the help of Theorems 5.1 and 6.1 the following result.

**Theorem 7.1.** Suppose that $\delta_1 = \delta_2 = \delta_3 = 0$. Then the following statements hold.

1. The resource and intermediate consumer-free periodic solution $(0, 0, z_{d_2}^*(t))$ is unstable.
2. The intermediate consumer-free periodic solution $(r/a, 0, z_{d_2}^*)$ is stable provided that

$$k_1 g_1(r/a - d_1)T < g_2^*(0)\mu/d_2.$$
and obtain the following scaled system

\[(k_1 g_1(r/a) - d_1) T < c_{a_1} \mu / d_2.\]

3. The intermediate consumer-free periodic solution \((r/a, 0, z^*_{d_2})\) is unstable provided that

\[(k_1 g_1(r/a) - d_1) T > g'_2(0) \mu / d_2.\]

It is now easy to see that a voracious top predator can always stabilize the system, driving the intermediate consumer to extinction and the prey to the carrying capacity of the environment. Also, for \(\mu\) large enough or \(T\) small enough, the global stability condition is always satisfied. Note that, for a significant part of the parameter space, the dynamical outcome does not depend upon the initial population sizes, which is perhaps not surprising, having in view that we study a model with predator-dependent functional responses, as opposed to a model with ratio-dependent functional responses.

We may further particularize \(g_i(x) = (a_i x) / (1 + b_i x),\) \(i \in \{1, 2\},\) and obtain that \((r/a, 0, z^*_{d_2})\) is stable provided that \(T < (a_2 \mu (a + b_1 r)) / (d_2 (k_1 a_1 r - d_1 a - d_1 b_1 r))\) and unstable provided that the reverse inequality holds, that is, a result similar to Theorem 3.1 in Zhang and Chen [25].

In the situations in which the intermediate consumer-free equilibrium is globally asymptotically stable, or at least the intermediate consumer population stabilizes below the economic injury level, it would be interesting from a practical point of view to give a general estimate of the time required for the intermediate consumer population to drop below the economic injury level. Unfortunately, we were not able to address this issue in this work.

8. Numerical simulations. We are now concerned with the numerical investigation of some situations not covered by our Theorems 5.1 and 6.1 which may lead to a chaotic behavior of the system. Following Klebanoff and Hastings [11] and Kuznetsov and Rinaldi [12], we rescale the variables using the formulas

\[x_1 = \frac{ax}{r}, \quad x_2 = \frac{ay}{rk_1}, \quad x_3 = \frac{az}{rk_1k_2}, \quad s = rt\]

and obtain the following scaled system

(SC)

\[
\begin{align*}
x'_1(s) &= x_1(s)[1 - x_1(s)] - \frac{m_1 x_1(s)}{1 + n_1 x_1(t)} x_2(s), \quad s \neq (n + l - 1)T_1, \quad s \neq nT_1; \\
x'_2(s) &= \frac{m_1 x_1(s)}{1 + n_1 x_1(t)} x_2(s) - \frac{m_2 x_2(s)}{1 + n_2 x_2(s)} x_3(s), \quad s \neq (n + l - 1)T_1, \quad s \neq nT_1; \\
x'_3(s) &= \frac{m_2 x_2(s)}{1 + n_2 x_2(t)} x_3(s) - D_1 x_1(s), \quad s \neq (n + l - 1)T_1, \quad s \neq nT_1; \\
\Delta x_1(s) &= -\delta_1 x_1(s), \quad s = (n + l - 1)T_1; \\
\Delta x_2(s) &= -\delta_2 x_2(s), \quad s = (n + l - 1)T_1; \\
\Delta x_3(s) &= -\delta_3 x_3(s), \quad s = (n + l - 1)T_1; \\
\Delta x_1(s) &= 0, \quad s = nT_1; \\
\Delta x_2(s) &= 0, \quad s = nT_1; \\
\Delta x_3(s) &= \mu_1, \quad s = nT_1.
\end{align*}
\]
where
\[
\begin{align*}
m_1 &= \frac{a_1 k_1}{a}, \quad n_1 = \frac{b_1 r}{a}, \quad n_2 = \frac{r b_2 k_1}{a}, \quad D_1 = \frac{d_1}{r}, \quad D_2 = \frac{d_2}{r}, \quad T_1 = rT, \\
\mu_1 &= \frac{a \mu}{r k_1 k_2}.
\end{align*}
\]

It is easy to see that the corresponding unperturbed system (RSC)
\[
\begin{align*}
x'_1(s) &= x_1(s)[1 - x_1(s)] - \frac{m_1 x_1(s)}{1 + n_1 x_1(s)} x_2(s), \quad s \neq (n + l - 1)T_1, \ s \neq nT_1; \\
x'_2(s) &= \frac{m_1 x_1(s)}{1 + n_1 x_1(t)} x_2(s) - \frac{m_2 x_2(s)}{1 + n_2 x_2(s)} x_3(s) - D_1 x_2(s), \quad s \neq (n + l - 1)T_1, \ s \neq nT_1; \\
x'_3(s) &= \frac{m_2 x_2(s)}{1 + n_2 x_2(t)} x_3(s) - D_2 x_3(s), \quad s \neq (n + l - 1)T_1, \ s \neq nT_1;
\end{align*}
\]
has at most five equilibria, namely
1. The trivial equilibrium \( O = (0, 0, 0) \).
2. The intermediate consumer and top predator-free equilibrium \( R = (1, 0, 0) \).
3. The top predator-free equilibrium
\[
RC = (D_1/(m_1 - n_1 D_1), (m_1 - n_1 D_1 - D_1)/(m_1 - n_1 D_1)^2, 0).
\]
4. The positive equilibria
\[
P_i = (x_{i1}^{P1}, D_2/(m_2 - n_2 D_2), x_3^{P1}), \quad P_2 = (x_{i2}^{P2}, D_2/(m_2 - n_2 D_2), x_3^{P2}),
\]
where
\[
\begin{align*}
x_{i1}^{P1} &= \frac{n_1 - 1}{2n_1} + (-1)^i \sqrt{(n_1 + 1)^2 - 4 \frac{m_1 n_1 D_2}{m_2 - n_2 D_2}}, \\
x_{3}^{P1} &= \frac{1}{m_2 - n_2 D_2} \left( \frac{m_1 x_{i1}^{P1}}{1 + n_1 x_{i1}^{P1}} - D_1 \right), \quad i \in \{1, 2\}.
\end{align*}
\]
Note that the first two equilibria exist irrespective of the values of the parameters which characterize the system, while several conditions need to be satisfied for the existence of the last three equilibria.

The dynamics of the unperturbed system (RSC) has been studied in detail by Klebanoff and Hastings in [11] and by Kuznetsov and Rinaldi in [12]. However, the behavior of the perturbed system (SC) is severely affected by our periodic forcing and the qualitative picture bears little resemblance, at least for significant forcing, to that of the unperturbed system.

From Theorem 5.1, it is easy to see that the intermediate consumer-free periodic solution is unstable provided that \( m_2 < m_{2s} \), where
\[
m_{2s} = \frac{\ln(1 - \delta_2) + \int_{0}^{T_1} \frac{m_1(x_1)^{(s)}(s)}{1 + n_1(x_1)^{(s)}(s)} ds - D_1 T_1}{\int_{0}^{T_1} (x_3)^{(s)}(s) ds}
\]
and locally stable provided that the reverse inequality is satisfied.
For $m_1 = 2.4$, $n_1 = 3$, $m_2 = 0.02$, $n_2 = 0.4$, $D_1 = 0.4$, $D_2 = 0.01$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\delta_3 = 0.5$, $\mu_1 = 0.25$, $T_1 = 10$, $l = 0.5$ (part of the values are close to the ones used by McCann and Yodzis in [19]) and $x_1(0) = 0.75$, $x_2(0) = 0.49$, $x_3(0) = 0.05$, it is seen that the intermediate consumer-free periodic solution is unstable and the stabilizing value is $m_{2s} = 0.098$. The unperturbed system has a top predator-free equilibrium, but no positive equilibria.

It is then seen that in this case the trajectory of the perturbed system tends to a periodic orbit of period $T_1$. Apart from deciding the stability or instability of the intermediate consumer-free periodic solution, the parameter $m_2$ does not seem to otherwise influence the qualitative properties or the shape of the limiting set. The behavior of the trajectory is depicted in Figure 8.1.

**Fig. 8.1.** $m_1 = 2.4$, $n_1 = 3$, $m_2 = 0.02$, $n_2 = 0.4$, $D_1 = 0.4$, $D_2 = 0.01$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\delta_3 = 0.05$, $\mu_1 = 0.25$, $T_1 = 10$, $l = 0.5$. The trajectory approaches a periodic orbit of period $T_1$. The unperturbed system has a top predator-free equilibrium, but no positive equilibria.

The intermediate consumer-free periodic solution is unstable and the stabilizing value is $m_{2s} = 1.329$. The unperturbed system has a top predator-free equilibrium and a positive equilibrium.

A related behavior is captured in Figure 8.2 for $m_1 = 10$, $n_1 = 3$, $D_1 = 0.4$, $m_2 = 0.1$, $n_2 = 0.4$, $D_2 = 0.01$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\delta_3 = 0.05$, $\mu_1 = 0.25$, $T_1 = 11$, $l = 0.5$ and $x_1(0) = 0.75$, $x_2(0) = 0.49$, $x_3(0) = 0.05$. The intermediate consumer-free periodic solution is unstable and the stabilizing value is $m_{2s} = 1.824$. In this case the trajectory of the perturbed system tends to a periodic orbit of period $3T_1$.

A typical example of chaotic behavior (strange attractor) is captured in Figure 8.3 for $m_1 = 10$, $n_1 = 3$, $D_1 = 0.4$, $m_2 = 0.1$, $n_2 = 0.4$, $D_2 = 0.01$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\delta_3 = 0.05$, $\mu = 0.25$, $T_1 = 30$, $l = 0.5$ and $x_1(0) = 0.75$, $x_2(0) = 0.49$, $x_3(0) = 0.05$. The intermediate consumer-free periodic solution is unstable and the stabilizing value is $m_{2s} = 1.424$. Again, the unperturbed system has a top predator-free equilibrium and a positive equilibrium. The two dimensional plot $x_2$ vs. $x_1$ and the time series for $x_1$, $x_2$, $x_3$ also indicate that the trajectory has a chaotic behavior. A slight increase in $m_2$ ($m_2 = 0.109$) “stabilizes” the behavior of the system, and the trajectory tends again to a periodic solution of period $T_1$. 
Fig. 8.2. $m_1 = 10, n_1 = 3, D_1 = 0.4, m_2 = 0.1, n_2 = 0.4, D_2 = 0.01, \delta_1 = 0.1, \delta_2 = 0.3, \delta_3 = 0.05, \mu = 0.25, T_1 = 11, l = 0.5$. The trajectory approaches a periodic orbit of period $3T_1$. The unperturbed system admits a top predator-free equilibrium and a positive equilibrium.

Fig. 8.3. $m_1 = 10, n_1 = 3, D_1 = 0.4, m_2 = 0.1, n_2 = 0.4, D_2 = 0.01, \delta_1 = 0.1, \delta_2 = 0.3, \delta_3 = 0.05, \mu = 0.25, T_1 = 30, l = 0.5$. The trajectory is chaotic (bistability-like scenario). The unperturbed system admits a top predator-free equilibrium and a positive equilibrium.

A somewhat similar situation is captured in Figure 8.4 for $m_1 = 10, n_1 = 2, D_1 = 0.4, m_2 = 0.1, n_2 = 0.4, D_2 = 0.01, \delta_1 = 0.1, \delta_2 = 0.3, \delta_3 = 0.05, \mu = 0.25, T_1 = 10, l = 0.5$ and $x_1(0) = 0.75, x_2(0) = 0.49, x_3(0) = 0.05$. The intermediate consumer-free periodic solution is unstable and the stabilizing value is $m_{2s} = 1.745$. The unperturbed system has a top predator-free equilibrium and two positive equilibria.
A slight increase in $m_2$ ($m_2 = 0.1119$) “stabilizes” the behavior of the system, and the trajectory tends again to a periodic solution of period $T_1$. That is, $m_2$ does not have only the potential to stabilize the intermediate consumer-free periodic solution, but also the potential to mitigate the chaotic behavior of a trajectory for certain values significantly smaller than the stabilizing critical value, an increase of $m_2$ over these values ensuring that the trajectories of the system tend to certain periodic solutions.

**Fig. 8.4.** $m_1 = 10$, $n_1 = 2$, $D_1 = 0.4$, $m_2 = 0.1$, $n_2 = 0.4$, $D_2 = 0.01$, $\delta_1 = 0.1$, $\delta_2 = 0.3$, $\delta_3 = 0.05$, $\mu = 0.25$, $T_1 = 10$, $l = 0.5$. The trajectory is chaotic (bistability-like scenario). The unperturbed system admits a top predator-free equilibrium and two positive equilibria.

9. **Concluding remarks.** In this paper, an integrated pest management model described through an impulsively perturbed tritrophic simple food chain system is proposed and investigated. To control the behavior of the system, biological controls, in the form of periodic release of top predators in a fixed amount and chemical controls, in the form of periodic pesticide spraying, are employed. It is assumed that as a result of pesticide spraying fixed proportions of resource biomass, intermediate consumer biomass and top predator biomass are degraded each time. Nonlinear general smooth functions are used to model the functional response of the intermediate consumer and of the top predator and a general prey-dependent model is consequently obtained.

By means of the Floquet theory of impulsively perturbed systems of ordinary differential equations, it is seen that the local stability of the intermediate consumer-free periodic solution is governed by a threshold-like inequality, provided that a certain condition on the productivity of the resource is satisfied. If the reverse of the productivity condition is satisfied, then the resource and intermediate consumer-free periodic solution is globally asymptotically stable.

A sufficient condition for the global stability of the intermediate consumer-free periodic solution, corresponding to the ultimate success of our pest management strategy, is established, while it is observed that, biologically speaking, the integrated pest management strategy can be considered successful when the intermediate consumer
population stabilizes under a certain economic injury level, not necessarily when it is completely eradicated. Formally, both the local and global stability condition display a significant dependence on the functional response of the top predator.

It is observed that, theoretically speaking, the control strategy can be always made to succeed by the use of proper pesticides, while as far as the biological control is concerned, its global effectiveness can also be reached provided that the top predator is voracious enough, or the amount $\mu$ of top predator released each time is large enough or the period $T$ is small enough. Any of these features alone can ensure the global success of our control strategy, although in concrete situations these may or may not be biologically feasible or may require a large amount of resources.

Finally, a numerical analysis of some situations leading to a chaotic behavior of the system is also provided.

REFERENCES