A new type of edge-derived vertex coloring

Ervin Győri *

Hungarian Academy of Sciences, Alfréd Rényi Institute of Mathematics, P.O.B. 127, Budapest H-1364, Hungary

Cory Palmer

Central European University, Department of Mathematics and its Applications, Nádor u. 9, Budapest H-1051, Hungary

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Abstract

We study the minimum number of weights assigned to the edges of a graph $G$ with no component $K_2$ so that any two adjacent vertices have distinct sets of weights on their incident edges. The best possible upper bound on this parameter is proved.

Keywords: Edge coloring; Chromatic number; Edge-weighting

1 Introduction

All graphs we discuss are simple (note that most results hold for multigraphs too) and finite. Let $G$ be a graph and $k$ a non-negative integer. A $k$-edge-weighting of $G$ is a mapping $\varphi : E(G) \rightarrow \{1, 2, \ldots, k\}$. (In this paper the term “weighting” will always refer to edges while “coloring” will always refer to vertices.) The weight set (with respect to $\varphi$) of a vertex $x \in V(G)$ is the set $S_{\varphi}(x)$ of weights of edges incident to $x$ (the subscript $\varphi$ can be omitted when it does not cause confusion). A $k$-edge-weighting $\varphi$ is vertex-coloring by sets if $S_{\varphi}(x) \neq S_{\varphi}(y)$ whenever vertices $x$, $y$ are adjacent (typically we will

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omit the phrase “by sets”). For a graph $G$ we are interested in the minimum $k$ such that there exists a $k$-edge-weighting of $G$ that is vertex-coloring. If $G$ has a component $K_2$, then $G$ cannot have a vertex-coloring edge-weighting, so we (have to) assume that $G$ has no such component. If $G$ is a graph with components $G_1, \ldots, G_n$, then we can take the maximum of these minima componentwise, so the analysis of the vertex-coloring edge-weightings can be restricted to connected graphs. Therefore all graphs will be assumed to be connected unless otherwise stated.

In this paper we will study $k$-edge-weightings that are vertex-coloring by sets. Actually, different kinds of edge-weightings deriving proper vertex-colorings have been studied. We say that a $k$-edge-weighting of $G$ is vertex-coloring by sums if for every edge $xy$ in $G$, the sum of the weights appearing on the edges incident to $x$ is distinct from the sum of the weights appearing on the edges incident to $y$. Similarly, we say that a $k$-edge-weighting of $G$ is vertex-coloring by multisets if for every edge $xy$ in $G$, the multiset of the weights appearing on the edges incident to $x$ is distinct from the multiset of the weights appearing on the edges incident to $y$. Considering these related concepts without appropriate notation, we suggest to introduce coherent notation as follows. Let $\chi_{ew}(G)$ denote the minimum $k$ such that there is a $k$-edge-weighting of $G$ that vertex-colors $G$ by the sums of weights of the edges incident to the vertex, $\chi_{em}(G)$ the minimum $k$ such that there is a $k$-edge-weighting of $G$ that vertex-colors $G$ by multisets of weights (of the edges incident to the vertex) and finally $\chi_{es}(G)$ the minimum $k$ such that there is a $k$-edge-weighting of $G$ that vertex-colors $G$ by sets of weights (of the edges incident to the vertex). (The last one was denoted by $\text{gndi}(G)$ in [1], and called the general neighbor distinguishing index of $G$, but we think it should be fit into this more coherent terminology and notation.)

Recently, Addario-Berry et al. [3] showed that for every graph $G$ without an edge component, $\chi_{em}(G) \leq 4$, i.e., it has a 4-edge-weighting that is vertex-coloring by multisets and if the minimum degree is at least 1000 then $\chi_{em}(G) \leq 3$. Furthermore, Addario-Berry et al. [2] showed that for every graph $G$ without an edge component, $\chi_{ew}(G) \leq 30$, i.e., it has a 30-edge-weighting that is vertex-colorings by sums.

The problem of determining $\chi_{es}(G)$ is deeper than it first seems to be: it is easy to see that to decide if $\chi_{es}(G) = 2$ for a bipartite graph $G$ is equivalent to decide if the hypergraph of the neighborhoods of the vertices in one class of the bipartition is 2-colorable (has the B-property in other words). In [1],

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Györi et al. prove that if $G$ is graph without an edge component then

$$\chi_e^c(G) \leq 2\lceil \log_2 \chi(G) \rceil + 1.$$  

The following theorem, also proved in [1], will assist us later in the proof of the main result. For an edge-weighting $\varphi$ and for a set $X$ of vertices, let $S_\varphi(X)$ denote the family of all weight-sets of vertices of $X$, i.e., $S_\varphi(X) = \{S_\varphi(x) \mid x \in X\}$.

**Theorem 1.** If $G$ is a bipartite graph without an edge component, then $\chi_e^c(G) \leq 3$. Furthermore, there is an edge-weighting $\varphi$ and bipartite classes $X,Y$ of $G$ such that,

$$S_\varphi(X) \subseteq \{\{1,2\}, \{3\}\},$$
$$S_\varphi(Y) \subseteq \{\{1\}, \{2\}, \{1,3\}, \{2,3\}\}.$$  

The main result of this paper is as follows.

**Theorem 2.** If $G$ is graph without an edge component and $\chi(G) \geq 3$ then

$$\chi_e^c(G) = \lceil \log_2 \chi(G) \rceil + 1.$$  

In [1] it is asked whether there exists a planar graph $G$ with $\chi_e^c(G) > 3$. Theorem 1 combined with Theorem 2 and the Four Color Theorem imply that there is no such graph.

**Corollary 3.** If $G$ is a planar graph without an edge component then $\chi_e^c(G) \leq 3$.  

The proof of Theorem 2 will be separated into three parts. First we prove it for $\chi(G) \leq 4$, then for $5 \leq \chi(G) \leq 8$, and finally for $\chi(G) \geq 8$. The next section will be concerned with the proof of the upper bound. The lower bound is a simple observation and will be used implicitly in the proofs in Section 2.

**Remark 4.** If $G$ is a graph then $\chi_e^c(G) \geq \lceil \log_2 \chi(G) \rceil + 1$. 

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Proof. Assume that we have a vertex-coloring edge-weighting of $G$ with $k = \chi_s(G)$ weights, and so we have at most $2^k$ different weight sets appearing in $G$. This naturally gives us a proper vertex-coloring of $G$ with $2^k$ colors. However, it is clear that a vertex with weight set $S$ and a vertex with weight set $\{1, 2, \ldots, k\} - S$ cannot be neighbors as the weight sets of neighbors must have a nonempty intersection (the weight of the edge connecting neighbors is necessarily in the intersection of their weight sets). Therefore we can color such vertices with the same color and thus at most $2^k - 1$ different colors are needed to color $G$. So, $\chi(G) \leq 2^{k-1}$ yields $\lceil \log_2 \chi(G) \rceil \leq k - 1$. 

The parameter $\chi^e_s(G)$ is not a monotone graph parameter under the addition of edges. For example, the path on 4 vertices, $P_4$, has $\chi^e_s(P_4) = 3$ but the cycle on 4 vertices, $C_4$, has $\chi^e_s(C_4) = 2$. However, our results imply that for graphs of chromatic number at least 3 the parameter $\chi^e_s(G)$ is in fact monotone. It is somewhat surprising that monotonicity seems to be difficult to prove directly. Let us refer to the fact that the proof for 3-chromatic graphs is considerably more difficult than the proof for 4-chromatic graphs.

2 Proof of Theorem 2

Additional notation used in the proof is mostly standard. In particular, the neighbors of a vertex $v$ are denoted by $N(v)$. Similarly, the set of all neighbors of a set of vertices $X$ is denoted by $N(X) = \bigcup_{v \in X} N(v) - X$. The set of all edges between two disjoint sets of vertices $X$ and $Y$ is denoted by $E(X, Y)$. With a slight abuse of notation $E(v, X)$ is the set of all edges between vertex $v$ and the set of vertices $X$. We will call an edge-weighting of $G$ canonical if there is a proper coloring (with $\chi(G)$ colors) of the vertices such that the weight sets appearing in one color class are strictly disjoint from those appearing in another. Note that a canonical edge-weighting is necessarily vertex-coloring (but a vertex-coloring edge-weighting need not be canonical).

Theorem 2 will follow from three lemmas.

**Lemma 5.** If $G$ is graph without an edge component and $3 \leq \chi(G) \leq 4$, then

$$\chi^e_s(G) = 3.$$
Lemma 6. If $G$ is a graph without an edge component and $5 \leq \chi(G) \leq 8$, then
\[ \chi^e_s(G) = 4. \]

Lemma 7. If $G$ is a graph without an edge component and $\chi(G) \geq 8$, then
\[ \chi^e_s(G) = \lceil \log_2 \chi(G) \rceil + 1. \]

2.1 Proof of Lemma 5

Let $G$ be a 3-chromatic graph. We call a 3-coloring $X, Y, Z$ of $V(G)$ stable if the following three conditions hold:

(i) if $x \in X$ then $N(x) \cap Y \neq \emptyset$
(ii) if $y \in Y$ then $N(y) \cap Z \neq \emptyset$
(iii) if $z \in Z$ then $N(z) \cap X \neq \emptyset$

In other words, each vertex in $X$ must have a neighbor in $Y$, each vertex in $Y$ must have a neighbor in $Z$ and each vertex in $Z$ must have a neighbor in $X$. If a 3-coloring is not stable call it unstable and observe that it must contain a vertex that fails to satisfy the above requirement. We will call such a vertex unstable, otherwise a vertex is stable.

Proposition 8. If $G$ is 3-chromatic and has a stable 3-coloring $X, Y, Z$, then $\chi^e_s(G) = 3$. Furthermore, there is a canonical edge-weighting $\varphi$ such that,
\[ S_{\varphi}(X) \subseteq \{\{3\}, \{1, 3\}\}, \]
\[ S_{\varphi}(Y) \subseteq \{\{2\}, \{2, 3\}\}, \]
\[ S_{\varphi}(Z) \subseteq \{\{1\}, \{1, 2\}\}. \]

Proof. If $X, Y, Z$ is a stable 3-coloring of $G$ then weight all $E(X, Y)$ edges with 3, all $E(X, Z)$ edges with 1 and all $E(Y, Z)$ edges with 2. Notice that among vertices of $X$ the only possible weight sets are $\{3\}$ and $\{1, 3\}$, among vertices of $Y$ the only possible weight sets are $\{2\}$ and $\{2, 3\}$ and among vertices of $Z$ the only possible weight sets are $\{1\}$ and $\{1, 2\}$. Clearly this edge-weighting yields $\varphi$ with the desired properties. \qed

Therefore to show a 3-chromatic graph has $\chi^e_s(G) = 3$ it is sufficient to find a stable 3-coloring.

Proposition 9. If $G$ is 3-chromatic and contains a triangle, then $G$ has a stable 3-coloring.
Proof. Let $X,Y,Z$ be the color classes of an arbitrary 3-coloring of $G$. Assume $X,Y,Z$ is an unstable 3-coloring of $G$ otherwise we are done. We will attempt to “stabilize” the coloring by moving (recoloring) unstable vertices to different color classes so they become stable in the new 3-coloring of $G$.

First we move all unstable vertices in $X$ to $Y$ where they will be stable. Next, we will move the unstable vertices of $Y$ to $Z$ where they will be stable. Finally, we move the unstable vertices of $Z$ to $X$ where they will be stable. While performing these three steps new unstable vertices must appear otherwise we have a stable 3-coloring and we are done. Let us perform the above three steps repeatedly in succession. If, at some point during this process, we reach a stable 3-coloring then we are done. Because $G$ is finite this process must eventually repeat itself (if it never reaches a stable 3-coloring).

Therefore, we must eventually return to a previous assignment of colors to the vertices. Let us examine a vertex $v$ that at some step is in color class $X$ then moves to $Y$ and eventually returns to $X$. Because our “stabilizing” steps do not allow $v$ to move from $Y$ to $X$ directly, we must have a step where $v$ moves from $Y$ to $Z$ and eventually a step where $v$ moves from $Z$ to $X$. The only way this can occur is if the neighbors of $v$ also move from class to class, i.e. for $v$ to be moved from $X$ to $Y$ its neighbors must all be in $Z$, for $v$ to be moved from $Y$ to $Z$ all of its neighbors must be in $X$ and for $v$ to be moved from $Z$ to $X$ all of its neighbors must be in $Y$. So, if a vertex $v$ moves and eventually returns to a previous assignment then so must all of its neighbors (and so on). This means that for each vertex there is a step where it becomes unstable (as we only move a vertex when it is unstable). However, it is clear that all the vertices of a triangle are stable and cannot become unstable (in any proper coloring of $G$ each vertex of the triangle must be in distinct color classes). So, if $G$ contains a triangle the repeated recoloring of the vertices of $G$ must eventually stop with a stable 3-coloring.

A number of stable and unstable graphs are known. In particular, the triangle and the Petersen Graph are stable, while any odd cycle (longer than 3) is unstable.

We introduce the notion of an almost-canonical edge-weighting of a 3-chromatic graph to assist us in the proof of Theorem 2. We call an edge-weighting, $\varphi$, of a 3-chromatic graph almost-canonical if there exists a 3-coloring $X,Y,Z$ and a vertex $x \in X$ such that the collections of weight sets $S_\varphi(X - x), S_\varphi(Y), S_\varphi(Z)$ are disjoint and $S_\varphi(x) \not\subseteq S_\varphi(X)$. Note that an almost-canonical edge-weighting need not be vertex-coloring.
Proposition 10. If $G$ is an unstable 3-chromatic graph and $x$ is an arbitrary vertex of $G$, then $G$ has an almost-canonical 3-edge-weighting with 3-coloring $X,Y,Z$ with $x \in X$ such that,

$$S_\varphi(X - x) \subseteq \{\{3\}, \{1,3\}\},$$
$$S_\varphi(Y) \subseteq \{\{2\}, \{2,3\}\},$$
$$S_\varphi(Z) \subseteq \{\{1\}, \{1,2\}\},$$
$$S_\varphi(x) = \{1\}.$$

**Proof.** Let $x$ be an arbitrary vertex of $G$ as in the statement of the proposition. Let us add two new vertices $v$ and $w$ and three new edges $xv$, $xw$, and $vw$ to $G$ thus creating a triangle in this new graph. By Proposition 9, this new graph has a stable 3-coloring. Let $X,Y,Z$ be the color classes of this coloring after removing $v$ and $w$ (and their incident edges) and assume without loss of generality that $x \in X$. Because we only removed neighbors of $x$ from the stable graph, $x$ is the only unstable vertex in the coloring of $G$, i.e., $x$ has no neighbors in $Y$.

Now, weight all edges in $E(X,Y)$ with weight 3, all edges in $E(X,Z)$ with weight 1, and all edges in $E(Y,Z)$ with weight 2. Now, the only possible weight sets in $X - x$ are $\{3\}$ and $\{1,3\}$, the only possible weight sets in $Y$ are $\{2\}$ and $\{2,3\}$, and the only possible weight sets in $Z$ are $\{1\}$ and $\{1,2\}$ while $x$ must have weight set $\{1\}$. Clearly this edge-weighting is almost-canonical and yields $\varphi$ with the desired properties. 

Proposition 11. Let $G$ be a unstable 3-chromatic graph with a vertex of degree 1, then $\chi^*_e(G) = 3$.

**Proof.** Let $x$ be a vertex in of degree 1 in $G$. Call $z$ the single neighbor of $x$. The vertex $z$ had degree at least 2, otherwise $xz$ is an isolated edge. Let us add two new vertices $v$ and $w$ and three new edges $xv$, $xw$, and $vw$ to $G$ thus creating a triangle in this new graph. By Proposition 9, this new graph has a stable 3-coloring. Let $X,Y,Z$ be the color classes of this coloring after removing $v$ and $w$ (and their incident edges) and assume without loss of generality that $x \in X$. Because we only removed neighbors of $x$ from the stable graph, $x$ is the only unstable vertex in the coloring of $G$, i.e., $x$ has no neighbors in $Y$. Therefore $z \in Z$. If the neighbors of $z$ are all in $X$ then we can move $x$ to $Y$ thus making $x$ stable and keeping $z$ stable. This gives a stable 3-coloring of $G$, a contradiction. So, we may assume $z$ has a neighbor in $Y$. 

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Now, weight all edges in $E(X,Y)$ with weight $3$, all edges in $E(X,Z)$ with weight $1$, and all edges in $E(Y,Z)$ with weight $2$. The only possible weight sets in $X - x$ are $\{3\}$ and $\{1, 3\}$, the only possible weight sets in $Y$ are $\{2\}$ and $\{2, 3\}$, and the only possible weight sets in $Z$ are $\{1\}$ and $\{1, 2\}$ while $x$ must have weight set $\{1\}$. However, the only neighbor of $x$ is $z$ which has weight set $\{1, 2\}$. Therefore, this 3-edge-weighting is vertex-coloring.

**Proposition 12.** If $G$ is a 3-chromatic graph, then $\chi_e^*(G) = 3$.

**Proof.** By the previous propositions, we may assume that $G$ is triangle-free, the minimum degree of $G$ is at least 2 and $G$ does not have a stable 3-coloring. Let $x$ be an arbitrary vertex of $G$. Let us add two new vertices $v$ and $w$ and three new edges $xv$, $xw$, and $vw$ to $G$ thus creating a triangle in this new graph. By Proposition 9, this new graph has a stable 3-coloring. Let $X, Y, Z$ be the color classes of this coloring after removing $v$ and $w$ (and their incident edges) and assume without loss of generality that $x \in X$. Because we only removed neighbors of $x$ from the stable graph, $x$ is the only unstable vertex in the coloring of $G$, i.e., $x$ has no neighbors in $Y$. Now we will construct a vertex-coloring weighting of $G$. Let $L \subset N(x) \subset Z$ be the set neighbors of $x$ that themselves have no neighbors in $Y$ (note that each vertex in $L$ must have at least one neighbor other than $x$). Let $M = N(L) - x \subset X$ be the neighbors of $L$ (they are necessarily in $X$) excluding $x$.

We begin the weighting of $G$ as follows:

1. Weight all edges incident to $x$ with weight 1.

2. For every vertex $m \in M$ that has neighbor in $Z - L$, weight all edges in $E(m, L)$ with weight 2 (in step 6 all edges in $E(m, Z - L)$ will get weight 1).

3. For every vertex $m \in M$ with a neighbor $l \in L$ such that $l$ is incident to an edge with weight 2 (from step 2 or 3), weight edge $ml$ with 1 and all remaining $E(m, L)$ edges with 3.

4. Repeat step 3 until no such edge $m$ remains (this process will stop as $M$ is finite).

5. Weight all remaining unweighted edges in $E(M, L)$ with weight 3.
6. Weight all remaining unweighted edges in $E(X, Z)$ with weight 1, weight all edges in $E(X, Y)$ with weight 3 and weight all edges in $E(Y, Z)$ with weight 2.

Call this edge-weighting $\varphi$. Now let us verify that $\varphi$ is vertex-coloring. Consider a vertex $u \in G$. We distinguish three cases depending on the color of $u$.

Case 1. $u \in Z$.

(a) $S_\varphi(u) = \{1, 2\}$. The vertex $u$ is trivially distinguished from all vertices in $Y$ as they may only have weight sets $\{3\}$ or $\{2, 3\}$. The vertex $u$ is trivially distinguished from all vertices in $X$ as all weight sets in $X$ contain the weight 3 with the exception of the vertex $x \in X$ with the weight set $\{1\}$.

(b) $S_\varphi(u) = \{1\}$. The vertex $u$ is trivially distinguished from all neighbors as the weight set $\{1\}$ only appears in $Z$ with the exception of the vertex $x \in X$. However, by construction, $x$ will not be adjacent to any vertices of $Z$ with weight set $\{1\}$.

(c) $S_\varphi(u) = \{1, 3\}$. In $Z$ this weight set only appears in $L$. By the construction of our weighting it implies that all of the neighbors of $u$ in $M$ have no neighbors in $L$ with weight set $\{1, 2\}$ and no neighbors at all in $Z - L$. This means that the edges between $u$ and $M$ were weighted with 3 and, so the weight set of all neighbors of $u$ in $M$ is $\{3\}$.

Case 2. $u \in X$.

(a) $S_\varphi(u) = \{1, 3\}$. By construction the only vertices in $Z$ incident to $u$ have weight set $\{1, 2\}$ or $\{1\}$. So $u$ is distinguished from all of $Z$. The weight set $\{1, 3\}$ does not appear in $Y$ so $u$ is distinguished from all of $Y$.

(b) $S_\varphi(u) = \{3\}$. The weight set $\{3\}$ only appears in $X$, so $u$ is trivially distinguished from all neighbors.

(c) $S_\varphi(u) = \{1, 2, 3\}$. The weight set $\{1, 2, 3\}$ only appears in $X$, so $u$ is trivially distinguished from all neighbors.

(d) $S_\varphi(u) = \{1\}$. This only occurs if $u = x$. As in case 1.2 we observe that $x$ is not adjacent to any vertices of $Z$ with weight set $\{1\}$.
Case 3. $u \in Y$.

The only weight sets that appear in $Y$ are $\{2, 3\}$ and $\{2\}$ which cannot appear elsewhere, so $u$ is distinguished from all of its neighbors.

Therefore $\varphi$ is a vertex-coloring 3-edge-coloring, so $\chi_s^e(G) = 3$. \hfill $\Box$

We can use a similar technique as in the proof of Proposition 9 to show that 4-chromatic graphs have $\chi_s^e(G) = 3$.

**Proposition 13.** If $G$ is a 4-chromatic graph, then $\chi_s^e(G) = 3$. Furthermore, there is a canonical edge-weighting $\varphi$ with 4-coloring $X, Y, Z, W$ such that,

$$
S_\varphi(X) \subseteq \{\{3\}, \{1, 3\}\}, \\
S_\varphi(Y) \subseteq \{\{2\}, \{2, 3\}\}, \\
S_\varphi(Z) \subseteq \{\{1\}, \{1, 2\}\}, \\
S_\varphi(Z) = \{\{1, 2, 3\}\}.
$$

**Proof.** Without loss of generality assume $G$ is connected. Let $X, Y, Z, W$ be a 4-coloring of $G$ and assume $|W|$ is of minimal size, i.e., each vertex of $W$ has a neighbor in the other three color class. For the proof we will use a similar notion of an unstable vertex in a 3-chromatic graph. In particular we will call a vertex $v \in G$ unstable if it satisfies one of the three following conditions:

(i) $v \in X$ and $N(v) \cap Y = \emptyset$ and $N(v) \cap W = \emptyset$
(ii) $v \in Y$ and $N(v) \cap Z = \emptyset$ and $N(v) \cap W = \emptyset$
(iii) $v \in Z$ and $N(v) \cap X = \emptyset$ and $N(v) \cap W = \emptyset$

Now, we will attempt to “stabilize” the unstable vertices of $G$ by the following three steps.

1. If $v \in X$ is unstable then move $v$ to $Y$. Thus $v$ becomes stable.
2. If $v \in Y$ is unstable then move $v$ to $Z$. Thus $v$ becomes stable.
3. If $v \in Z$ is unstable then move $v$ to $X$. Thus $v$ becomes stable.

Let us repeat these three steps until the process stops or repeats itself. If the process repeats itself we know from the proof of Proposition 9 that there is a vertex $v$ that returns to a previous assignment after appearing in each color class $X, Y, Z$. As a result, the neighbors of $v$ will also complete a “circuit” around $X, Y, Z$ (and so on). So, we have a component of $G$ that
cannot be connected to any vertex in $W$. This contradicts the fact that $G$ is connected and 4-chromatic. So, the “stabilizing” process must eventually stop. When it does we have the following vertex-coloring weighting of $G$: weight $E(X,Y)$ edges with 3, weight $E(X,Z)$ edges with 1, weight $E(Y,Z)$ edges with 2, weight $E(X,W)$ edges with 3, weight $E(Y,W)$ edges with 2 and weight $E(Z,W)$ edges with weight 1. This edge-weighting will give the following arrangement of weight sets: In $X$ we have \{3\} and \{1, 3\}, in $Y$ we have \{2\} and \{2, 3\}, in $Z$ we have \{1\} and \{1, 2\}, in $W$ we have \{1, 2, 3\}. This gives the desired edge-weighting $\varphi$ which is clearly canonical and hence vertex-coloring. Therefore, $\chi^e_s(G) = 3$. \hfill $\Box$

Clearly, Propositions 12 and 13 imply Lemma 5.

2.2 Proof of Lemma 6

Before proving Lemma 6, we introduce some definitions. Let $X \subset V(G)$ be an independent set in $G$ (e.g. a color class in a coloring of $G$) and let $S_\varphi(X)$ be the set of weight sets appearing on vertices in $X$ under the vertex-coloring edge-weighting $\varphi$. Then $X$ is called $i$-safe (with respect to $\varphi$) if for any $S \in S_\varphi(X)$ we have $S \cup \{i\} \in S_\varphi(X)$. In particular, this implies that the addition of $i$ to any weight-set appearing on a vertex in $X$ does not change the property that $\varphi$ is vertex-coloring.

Additionally, $X$ is called $i$-free (with respect to $\varphi$) if $i \notin S$ for any $S \in S_\varphi(X)$, i.e., $i$ never appears in the weight set of any vertex $x \in X$. The following proposition implies Lemma 6 and will form the base case of the inductive proof of Lemma 7.

**Proposition 14.** If $G$ is a graph such that $5 \leq \chi(G) \leq 8$ then $\chi^e_s(G) = 4$. Furthermore, if $\chi(G) = 8$ then there is a vertex-coloring edge-weighting of $G$ and a coloring of $G$ with distinct color classes $X_1, X_2, X_3, X_4, Y$ such that $X_i$ is $i$-safe and $Y$ is $4$-free.

**Proof.** We explicitly construct an edge-weighting of $G$ where $\chi(G) = 8$. It will be clear from the proof that our weighting will also work for graphs of chromatic number between 5 and 8. Color the graph $G$ and let $H$ be the subgraph of $G$ induced by four color classes of $G$ and let $F = G[V(G) - V(H)]$ be the graph induced by the remaining color classes such that $V(F)$ is minimal, i.e., no vertex of $F$ can be colored with a color from $H$ (even after a recoloring of $H$).
The subgraph $H$ may not necessarily be connected. We will distinguish 5 types of components of $H$. Let $H_4$ be an arbitrary 4-chromatic component of $H$, let $H_3$ be an arbitrary 3-chromatic component of $H$, let $H_2 \neq K_2$ be an arbitrary bipartite component of $H$, let $xy$ be an arbitrary isolated edge of $H$ and let $v$ be an arbitrary isolated vertex of $H$. We will describe how to weight edges among these subgraphs. This technique should be followed for all such components.

By Proposition 13 we have a vertex-coloring edge-weighting, $\varphi_4$, of $H_4$ with $\chi_e(H_4) = 3$ weights. Let the color classes of $H_4$ be $X_4, Y_4, Z_4, W_4$ such that,

$$S_{\varphi_4}(W_4) = \{\{1, 2, 3\}\},$$
$$S_{\varphi_4}(X_4) \subseteq \{\{3\}, \{1, 3\}\},$$
$$S_{\varphi_4}(Y_4) \subseteq \{\{2\}, \{2, 3\}\},$$
$$S_{\varphi_4}(Z_4) \subseteq \{\{1\}, \{1, 2\}\}.$$

If $H_3$ has a stable 3-coloring then by Proposition 8 then we have a vertex-coloring edge-weighting, $\varphi_3$, of $H_3$ with $\chi_e(H_3) = 3$ weights. Let the color classes of $H_3$ be $X_3, Y_3, Z_3$ such that,

$$S_{\varphi_3}(X_3) \subseteq \{\{3\}, \{1, 3\}\},$$
$$S_{\varphi_3}(Y_3) \subseteq \{\{2\}, \{2, 3\}\},$$
$$S_{\varphi_3}(Z_3) \subseteq \{\{1\}, \{1, 2\}\}.$$

If $H_3$ does not have a stable 3-coloring then choose a vertex $u$ in $H_3$ that has a neighbor in $F$ (such an $u$ exists as $G$ is connected). Now, by Proposition 10 we have an almost-canonical edge-weighting, $\varphi_3$, of $H_3$ and a 3-coloring $X_3, Y_3, Z_3$ with $u \in X_3$ such that,

$$S_{\varphi_3}(X_3 - u) \subseteq \{\{3\}, \{1, 3\}\},$$
$$S_{\varphi_3}(Y_3) \subseteq \{\{2\}, \{2, 3\}\},$$
$$S_{\varphi_3}(Z_3) \subseteq \{\{1\}, \{1, 2\}\},$$
$$S_{\varphi_3}(u) = \{1\}.$$

Note that $u$ may have the same weight set as some of its neighbors in $Z_3$. (We will resolve this conflict later when weighting the edges from $u$ to $F$.)
By Theorem 1 we have a vertex-coloring edge-weighting, $\varphi_2$, of $H_2$ with $\chi^e_s(H_2) = 3$ weights and a bipartition $X_2, Y_2$ such that,

$$S_{\varphi_2}(X_2) \subseteq \{\{3\}, \{1, 2\}\},$$
$$S_{\varphi_2}(Y_2) \subseteq \{\{1\}, \{2\}, \{1, 3\}, \{2, 3\}\}.$$

Finally we will weight all isolated edges $xy$ with weight 2. We refer to this edge-weighting of all of the components of $H$ (and hence $H$ itself) as $\varphi$.

Now let:

$$X = X_4 \cup X_3 \cup X_2 \cup \{x\} \cup \{v\},$$
$$Y = Y_4 \cup Y_3 \cup Y_2 \cup \{y\},$$
$$Z = Z_4 \cup Z_3,$$
$$W = W_4.$$

By our choice of $H$, each vertex of $F$ has a neighbor in each of $X, Y, Z, W$. Let $A_1, A_2, A_3, A_4$ be the color classes of $F$ (if $\chi(G) < 8$ we just follow the given edge-weighting and ignore the steps involving the appropriate $A_i$). Let us assume the color classes of $F$ are colored such that a vertex in $A_i$ has a neighbor in each $A_j$ for $j < i$. Let us weight all edges $E(A_4, A_3)$ with weight 2. Let us weight all other edges in $F$ with weight 4.

Now it remains to weight all edges in $E(H, F)$. The table below describes how to weight edges between different color classes and shows what the possible weight sets are in each color class after weighting all of the edges of $G$. In particular, the first column represents each color class (some are split into distinct components). Recall that each vertex in $A_i$ has a neighbor in each of $X, Y, Z, W$. Columns two through five represent the weight to give edges between $A_i$ and the corresponding color class in a given row (when such an edge exists). We refer to the weighting of all of the edges of $G$ by $\psi$. The final column represents the possible weight sets appearing in the corresponding color class in a given row.
By examining the table, we can easily verify that \( \psi \) is vertex-coloring. Clearly the single weight set appearing in \( A_i \) (with respect to \( \psi \)) only appears in \( A_i \) for each \( i \). It remains to check that weight sets on vertices in \( H \) are distinct from those of their neighbors in \( H \) under the weighting \( \psi \). This is clear as for any independent set \( Q \in \{ X_4, Y_4, Z_4, W_4, X_3, Y_3, Z_3, Y_2, \{ x \}, \{ y \}, \{ v \} \} \) weight sets of \( Q \) (with respect to \( \psi \)) only modifies the weight set with respect to \( \phi \) by either adding weight 4 or adding a weight \( i \) where \( Q \) was \( i \)-safe (with respect to \( \phi \)) (neither of which can destroy the property of being vertex-coloring). In the case of \( X_2 \), we potentially added the weight 3 to the weight set \( \{ 1, 2 \} \). However, it is clear that no vertex of \( X_2 \) is adjacent to a vertex of \( H \) where the weight set \( \{ 1, 2, 3 \} \) appears. For a \( K_2 = xy \) in \( H \) we have \( S_f(x) \in \{ \{ 2 \}, \{ 2, 4 \} \} \) and \( S_f(y) \in \{ \{ 2 \}, \{ 2, 3 \} \} \) but both \( x \) and \( y \) cannot get weight set \( \{ 2 \} \) as \( G \) has no \( K_2 \) components, i.e., either \( x \) or \( y \) has a neighbor in \( F \) (and all edges from \( F \) to \( x \) or \( y \) get weight 3). Finally, for \( v \) is an isolated vertex in \( H \) will have weight set \( \{ 3 \} \) as \( v \) must have a neighbor in \( F \), but this weight set appears nowhere else in \( G \). Therefore our edge-weighting is vertex-coloring.

Furthermore, it is clear that when \( \chi(G) = 8 \) we have that \( A_2 \) is 1-safe, \( A_3 \) is 2-safe, \( A_4 \) is 3-safe, \( A_1 \) is 4-safe and \( W \) is 4-free thus giving the proposition.

\[ \square \]
2.3 Proof of Lemma 7

For technical reasons we prove the following stronger version of Lemma 7.

**Proposition 15.** Suppose $G$ is a graph such that $\chi(G) \geq 8$ and let $k = \lceil \log_2 \chi(G) \rceil$, then $\chi_s^e(G) \leq k + 1$. Furthermore, there exists a vertex-coloring $(k + 1)$-edge-weighting of $G$ and a coloring of $G$ with distinct color classes $X_1, X_2, \ldots, X_{k+1}, Y$ such that $X_i$ is $i$-safe and $Y$ is $(k + 1)$-free.

**Proof.** Let $G$ be a graph with chromatic number $\chi(G)$. There exists an integer $k$ such that $2^{k-1} < \chi(G) \leq 2^k$. We proceed by induction on $\chi(G)$. The base case $\chi(G) = 8$ holds by Proposition 14. So, let $\chi(G) > 8$ and assume the statement of the proposition for all graphs $H$ with $\chi(H) < \chi(G)$. Assume we have a coloring of $G$ with $\chi(G)$ colors where $H$ is the subgraph of $G$ induced by $2^k-1$ color classes and $F = G[V(G) - V(H)]$ is the subgraph induced by the remaining color classes such that $|V(F)|$ is minimal, i.e., no vertex in $F$ can be colored with a color in $H$ (even after any recoloring of $H$). By induction we have $\chi_s^e(H) = k$ and we have an $k$-edge-weighting of $H$ and a coloring of $H$ with distinct color classes $X_1, X_2, \ldots, X_k, Y$ such that $X_i$ is $i$-safe and $Y$ is $k$-free. Let us keep this edge-weighting of $H \subseteq G$ and weight the remaining edges of $G$.

First, weight all edges in $F$ with weight $k + 1$. Now it remains to weight the edges between $H$ and $F$. Label the color classes of $F$ with $(k-1)$-length binary strings from 0 to $\chi(G) - 2^{k-1}$. Each vertex $v$ in $F$ has a neighbor in each color class of $H$ (in particular each $X_i$ and $Y$). If the binary string corresponding to the color class of $v$ has an 1 in the $i$-th binary digit then weight all edges between $v$ and $X_i$ with weight $i$. Then weight all edges between $v$ and $Y$ with $k + 1$. Finally, for all remaining unweighted edges $vw \in E(F,H)$ we weight $vw$ as follows: if $w$ is incident to an edge with weight $k$ then weight $vw$ with weight $k$. Otherwise, weight $vw$ with weight $k + 1$. In this way, we guarantee that every weight set in $F$ has both of the weights $k$ and $k + 1$ while every weight set in $H$ has at most one of the weights $k$ and $k + 1$. Clearly, each color class in $F$ will have a unique weight set corresponding to its (unique) binary string. The color classes of $H$ were already distinguished by the first $k$ weights. The edges between $F$ and $H$ only added weight $k$ to $k$-safe color classes of $H$ or a new weight $k + 1$, so weight sets of neighbors in $H$ remain distinct. This gives a vertex-coloring $(k + 1)$-edge-weighting of $G$ where $k + 1 = \lceil \log_2 \chi(G) \rceil + 1$. Furthermore, for $i = 1, 2, \ldots, k-1$ class $X_i$ remains $i$-safe, the first class of $F$ (its corresponding
binary string is 00...0) is $k$-safe, class $Y$ is now $(k + 1)$-safe class and $X_k$ is $(k + 1)$-free.

\[\square\]

References

