Four parameter proximal point algorithms

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Abstract

Several strong convergence results involving two distinct four parameter proximal point algorithms are proved under different sets of assumptions on these parameters and the general condition that the error sequence converges to zero in norm. Thus our results address the two important problems related to the proximal point algorithm − one being of strong convergence (instead of weak convergence) and the other one being that of acceptable errors. One of the algorithms discussed was introduced by Y. Yao and M. A. Noor (2008) while the other one is new and it is a generalization of the regularization method initiated by N. Lehdili and A. Moudafi (1996) and later developed by H. K. Xu (2006). The new algorithm is also ideal for estimating the convergence rate of a sequence that approximates minimum values of certain functionals. Although these algorithms are distinct, it turns out that for a particular case, they are equivalent. Results of this paper extend and generalize several existing ones in the literature.

Keywords: Proximal point algorithm, regularization method, monotone operator, weak convergence, strong convergence, minimizer.

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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and Hilbertian norm $\| \cdot \|$. An important topic in nonlinear analysis and optimization concerns iterative methods for solving the following problem:

$$\text{find an } x \in D(A) \text{ such that } 0 \in A(x), \tag{1}$$

where $A : D(A) \subset H \to 2^H$ is a maximal monotone operator. By a monotone operator A, we mean one that satisfy the property

$$\langle x - x', y - y' \rangle \geq 0, \quad \forall\, (x, y), (x', y') \in A,$$

and saying that an operator is maximal monotone, we mean that it is maximal with respect to this (monotonicity) property. In other words, we mean that its graph, viewed as a monotone subset of $H \times H$, is not properly contained in the graph of any other monotone operator.

The classical method of solving problem (1) is the so called proximal point algorithm (PPA), which was introduced by Martinet [8] and later developed systematically by Rockafeller [10]. Since its birth in the early 70's, the PPA has enjoyed a lot of attention from several authors, (see, e.g., [3-5]), and it has since become a powerful and versatile technique for solving variational inequalities and many problems in convex optimization such as convex minimizations and convex-concave mini-max problems. Given any starting point x_0, the PPA according to Rockafellar generates a sequence $\{x_n\}$ conforming to the inclusion relation

$$x_n + e_n \in x_{n+1} + \beta_n A(x_{n+1}), \quad n = 0, 1 \ldots,$$

where $\{\beta_n\} \subset (0, \infty)$, $\{e_n\}$ is the sequence of errors and A is maximal monotone. Under summable errors, it was shown in [10] that the PPA converges weakly to a solution of problem (1). It is now known that the PPA fails to converge strongly in general, (see [4]), and for this reason much of research has been devoted to finding/formulating modified versions of the PPA to improve weak convergence to strong convergence. Although several researchers have achieved this task of strengthening weak convergence to strong convergence, they still used the fact that errors are summable − a condition too strong at least from the numerical point of view − to derive strong convergence results. Such results have prompted the current authors to construct an algorithm which still preserve its strong convergence under the general condition that errors converge to zero in norm, (see [1]). The current paper explores this weaker/relaxed condition on errors to derive strong convergence of two (distinct) generalized proximal point algorithms which agree or rather are equivalent for a special case. One of the algorithms (discussed in Section 3) was introduced by Yao and Noor [17], while the other (see Section 4) is new and it is an extension of the regularization method proposed by Xu [16]. Such a regularization method is in fact an extension of the so called prox-Tikhonov method (introduced by Lehdili and Moudafi [6]) as noted by Xu himself.

2 Preliminaries

We begin this section by giving a Lemma due to Xu. Most of the results of this paper are based on this Lemma. In section 4, we shall give an alternative proof of Lemma 6 below which is actually based on the arguments of the proof of this lemma. However, as opposed to the original proof of Lemma 6 (see [1]), this alternative proof makes use of the fact that the series on $\{\alpha_n\}$ is divergent.

Lemma 1 [15]. *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - a_n)s_n + a_n b_n + c_n, \quad n \geq 0,$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ satisfy the conditions: (i) $\{a_n\} \subset [0, 1]$, *with* $\sum_{n=0}^{\infty} a_n = \infty$, (ii) $\limsup_{n\to\infty} b_n \leq 0$, *and* (iii) $c_n \geq 0$ *for all* $n \geq 0$ *with* $\sum_{n=0}^{\infty} c_n < \infty$. *Then* $\lim_{n\to\infty} s_n = 0$.

The following well known lemma is also useful in the next sections.

Lemma 2 (Subdifferential Inequality).

$$\|x+y\|^2 \le \|y\|^2 + 2\langle x, x+y\rangle \quad \textit{for all} \quad x, y \in H.$$

The next two Lemmas are also well known. They can be found in many functional analysis books.

Lemma 3 (Resolvent Identity). *For any* $\beta, \gamma > 0$*, and* $x \in H$*, the identity*

$$J_\beta x = J_\gamma \left(\frac{\gamma}{\beta} x + \left(1 - \frac{\gamma}{\beta}\right) J_\beta x \right)$$

holds true, where $J_\beta = (I + \beta A)^{-1}$ *for a maximal monotone operator* $A : D(A) \subset H \to 2^H$*.*

Lemma 4 (Z. Opial, (see, e.g., [9], p. 5)). *Let F be a nonempty subset of H. Assume that the sequence $\{x_n\}$ satisfy the conditions (i)* $\lim_{n\to\infty} \|x_n - q\| = \rho(q)$ *exists for all $q \in F$, and (ii) any weak cluster point of $\{x_n\}$ belongs to F. Then, there exists a point $p \in F$ such that $\{x_n\}$ converges weakly to p.*

Lemma 5 [12]. *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a real Banach space and let $\{\rho_n\}$ be a sequence in* $[0,1]$, *with* $0 < \liminf_{n\to\infty} \rho_n \le \limsup_{n\to\infty} \rho_n < 1$*. Suppose* $x_{n+1} = \rho_n y_n + (1-\rho_n)x_n$ *for all integers* $n \ge 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$*. Then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$*.*

We now give a lemma that guarantees that $\{x_n\}$ generated by the iterative process (2) (see below) is bounded. For the case when $\lambda_n = 0$ for all n, the proof was given in [1] (see also [2]). The same proof can be extended for the general case when $\lambda_n \ne 0$ for all n, and it can also be applied to derive the boundedness of $\{v_n\}$ generated by algorithm (14) (see Section 4 below).

Lemma 6 *For any fixed $u, x_0 \in H$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \gamma_n J_{\beta_n} x_n + e_n, \quad n \ge 0, \tag{2}$$

where $J_{\beta_n} = (I + \beta_n A)^{-1}$*, for a maximal monotone operator* $A : D(A) \subset H \to 2^H$, $\beta_n > 0$, $\alpha_n \in (0,1)$*, and* $\lambda_n, \gamma_n \in [0,1]$ *with* $\alpha_n + \lambda_n + \gamma_n = 1$ *for all n. If either* $\sum_{n=0}^\infty \|e_n\| < \infty$ *or* $\{\|e_n\|/\alpha_n\}$ *is bounded, and* $F := A^{-1}(0) \ne \emptyset$*, then $\{x_n\}$ is bounded.*

3 A generalized proximal point algorithm

In this section, we discuss strong convergence of $\{x_n\}$ generated by algorithm (2) above under different sets of assumptions on $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$. The first result of this section makes use of the conditions

$$(C1) \lim_{n\to\infty} \left(\frac{1}{\beta_{n+1}} - \frac{1}{\beta_n} \right) = 0, \quad \text{and} \quad (C2) \sum_{n=0}^\infty \frac{|\alpha_{n+1} - \alpha_n|}{\beta_{n+1}} < \infty,$$

which were introduced in [2]. Some examples were given in [2] to show that in some cases these conditions hold true while the ones known before such as

$$\lim_{n\to\infty}(\beta_{n+1}-\beta_n)=0, \quad \text{and} \quad \sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty,$$

may fail. The latter condition was introduced in [14] while the former can be found in [12].

Concerning the sequence of errors $\{e_n\}$, we shall always assume that it satisfies either one of the following conditions

$$(E1)\ \sum_{n=0}^{\infty}\|e_n\|<\infty, \quad \text{or} \quad (E2)\ \lim_{n\to\infty}\frac{\|e_n\|}{\alpha_n}=0.$$

Rockafellar proved weak convergence of the PPA under the summability condition (E1) on errors, and since then this condition has been used extensively in proving either weak or strong convergence results of the PPA. It was in [1] that the condition (E2) was introduced and used to derive a strong convergence result of the modified PPA. We point out that these conditions are different. For example, $\|e_n\|=n^{-1}$ and $\alpha_n=1/\sqrt{n}$ satisfy (E2) but not (E1), while $\|e_n\|=n^{-2}$ and $\alpha_n=n^{-1}+(-1)^n(n+1)^{-1}$ satisfy (E1) but fails to satisfy (E2). The advantage of using condition (E1) over (E2) is that it allows one to choose freely the sequence of parameters $\{\alpha_n\}$. Despite that (E2) does not allow us such freedom, it is still a good condition since it covers the errors that are not summable, as shown in the above example. In fact, having any sequence of errors $\{e_n\}$, converging to zero in norm, one can always construct the PPA that is strongly convergent by constructing a sequence of parameters $\{\alpha_n\}$ in such a way that condition (E2) is satisfied (see Section 5). The resulting α_n's depend on $\{e_n\}$, but this is acceptable from the numerical point of view.

Theorem 1 *Assume that $A : D(A)\subset H\to 2^H$ is a maximal monotone operator and $F:=A^{-1}(0)\neq\emptyset$. Fix $u,x_0\in H$, and let $\{x_n\}$ be the sequence generated by algorithm* (2) *with the conditions:* (i) $\alpha_n\in(0,1)$ *with* $\lim_{n\to\infty}\alpha_n=0$, *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$, (ii) *either (E1) or (E2),* (iii) $\lambda_n,\gamma_n\in[0,1]$, $\alpha_n+\lambda_n+\gamma_n=1$ *with* $\sum_{n=0}^{\infty}\lambda_n<\infty$, *and* (iv) $\beta_n\in(0,\infty)$ *with* $\liminf_{n\to\infty}\beta_n>0$ *and (C1) being satisfied. If, in addition (C2) holds, then $\{x_n\}$ converges strongly to P_Fu, the projection of u on F.*

Proof: We know from Lemma 6 that $\{x_n\}$ is bounded. Denote

$$v_n:=\frac{x_{n+1}-\alpha_n u-\lambda_n x_n-e_n}{\gamma_n}. \tag{3}$$

Note that $\{v_n\}$ is bounded since $\{x_n\}$ is bounded and for $\alpha_n,\lambda_n\to 0$, we see that the weak ω-limit sets of $\{x_n\}$ and $\{v_n\}$ coincide, that is, $\omega_{\mathtt{w}}(\{x_n\})=\omega_{\mathtt{w}}(\{v_n\})$. Moreover, we have from (2), and (3),

$$Av_n\ni\frac{x_n-x_{n+1}+\alpha_n(u-x_n)+e_n}{\beta_n\gamma_n}. \tag{4}$$

Our aim is to prove that the relation $\omega_{\mathtt{w}}(\{x_n\}) \subset F$ holds, from which we can establish the inequality

$$\limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u\rangle \le 0. \tag{5}$$

Indeed, for some subsequence of $\{x_n\}$ converging weakly, to some $x_\infty \in F$, we have

$$\limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u\rangle = \lim_{k\to\infty} \langle u - P_F u, x_{n_k} - P_F u\rangle = \langle u - P_F u, x_\infty - P_F u\rangle \le 0.$$

In view of (4), it would be enough if we could show that

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\beta_n} = 0. \tag{6}$$

For this purpose, we first compare x_{n+2} and x_{n+1} as follows

$$\begin{aligned} x_{n+2} - x_{n+1} &= (1-\alpha_n)(J_{\beta_{n+1}} x_{n+1} - J_{\beta_n} x_n) + \lambda_{n+1}(x_{n+1} - J_{\beta_{n+1}} x_{n+1}) + \lambda_n (J_{\beta_n} x_n - x_n) \\ &+ (\alpha_{n+1} - \alpha_n)(u - J_{\beta_{n+1}} x_{n+1} + e_{n+1}/\alpha_{n+1}) + \alpha_n (e_{n+1}/\alpha_{n+1} - e_n/\alpha_n). \end{aligned}$$

Using the resolvent identity and the fact that the resolvent is nonexpansive, we get

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| &\le (1-\alpha_n) \left\| J_{\beta_{n+1}} x_{n+1} - J_{\beta_{n+1}} \left(\tfrac{\beta_{n+1}}{\beta_n} x_n + \left(1 - \tfrac{\beta_{n+1}}{\beta_n}\right) J_{\beta_n} x_n \right) \right\| \\ &+ (\lambda_n + \lambda_{n+1}) K + |\alpha_{n+1} - \alpha_n| M + \alpha_n \left\| \frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n} \right\| \\ &\le (1-\alpha_n) \left\| \tfrac{\beta_{n+1}}{\beta_n} (x_{n+1} - x_n) + \left(1 - \tfrac{\beta_{n+1}}{\beta_n}\right) (x_{n+1} - J_{\beta_n} x_n) \right\| \\ &+ (\lambda_n + \lambda_{n+1}) K + |\alpha_{n+1} - \alpha_n| M + \alpha_n \left\| \frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n} \right\| \\ &\le (1-\alpha_n) \tfrac{\beta_{n+1}}{\beta_n} \|x_{n+1} - x_n\| + \left| 1 - \tfrac{\beta_{n+1}}{\beta_n} \right| \|x_{n+1} - J_{\beta_n} x_n\| \\ &+ (\lambda_n + \lambda_{n+1}) K + |\alpha_{n+1} - \alpha_n| M + \alpha_n \left\| \frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n} \right\|, \end{aligned} \tag{7}$$

for some positive constants K and M. Note that we have from (2)

$$\|x_{n+1} - J_{\beta_n} x_n\| \le \alpha_n \|u - J_{\beta_n} x_n + e_n/\alpha_n\| + \lambda_n \|x_n - J_{\beta_n} x_n\|,$$

which together with (7) yield

$$\begin{aligned} \frac{\|x_{n+2} - x_{n+1}\|}{\beta_{n+1}} &\le (1-\alpha_n) \frac{\|x_{n+1} - x_n\|}{\beta_n} + \alpha_n \left| \frac{1}{\beta_{n+1}} - \frac{1}{\beta_n} \right| M + \frac{\alpha_n}{\beta_{n+1}} \left\| \frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n} \right\| \\ &+ \frac{|\alpha_{n+1} - \alpha_n|}{\beta_{n+1}} M + (2\lambda_n + \lambda_{n+1}) K', \end{aligned}$$

for some $K' > 0$. Equation (6) then follows from Lemma 1 and this last estimate.

The application of the subdifferential inequality to (2) yields

$$\|x_{n+1} - P_F u\|^2 \leq (1-\alpha_n)\|x_n - P_F u\|^2 + 2\alpha_n \langle u - P_F u + \frac{e_n}{\alpha_n}, x_{n+1} - P_F u\rangle. \tag{8}$$

Therefore from Lemma 1, inequality (5) and conditions (i) and (E2) of the theorem, we derive $x_n \to P_F u$. The proof is similar in the case when (E1) is satisfied. ■

Remark 1: Since λ_n is summable and $\{x_n\}$ is bounded, the term $\lambda_n(x_n - J_{\beta_n} x_n)$ can be regarded as the error term in the case when the sequence of errors is also summable. In that case, algorithm (2) assumes the form $x_{n+1} = \alpha_n u + (1-\alpha_n) J_{\beta_n} x_n + E_n$, and therefore can be analyzed in the same way we did in [2]. In particular, we reobtain Theorem 6 of [2] from Theorem 1, and other results of [2] can also be obtained in this way. From this point of view, Theorem 1 is already known. For the case when $\{e_n\}$ satisfy condition (E2), we can not absorb the term $\lambda_n(x_n - J_{\beta_n} x_n)$ into the error sequence, otherwise it will mean that the quotient $\|E_n\|/\alpha_n = \|\lambda_n(x_n - J_{\beta_n} x_n) + e_n\|/\alpha_n$ must always converge to zero. This is not the case as one can check by taking the sequences defined by $\lambda_n = n^{-2}$ and $\|e_n\| = \alpha_n^2$, with $\alpha_n = n^{-1} + (-1)^n (n+1)^{-1}$. In this case, our theorem remain valid, thus Theorem 1 is 'really' new. We mention here that the upper bound of $\|x_n\|$ is independent of the coefficients $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\gamma_n\}$. Even though the above quotient does not always converge to zero (as the above example show), $\|E_n\|$ does, so we may choose another sequence of parameters, say $\{\alpha_n^*\}$, to reobtain the condition $\|E_n\|/\alpha_n^* \to 0$. Of course such α_n^*'s result in another PPA of Halpern type − the form mentioned earlier − in which $\{E_n\}$ is the new error sequence, and again we may analyze the resulting algorithm as in [2]. The above theoretical observations illustrate the significance of the condition (E2).

Remark 2: Note that the series condition on $\{\lambda_n\}$ can be relaxed, but at the expense of stronger assumptions on $\{\beta_n\}$. For instance, we may assume that $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy

$$(C3) \; \sum_{n=0}^{\infty} \frac{|\lambda_{n+1} - \lambda_n|}{\beta_{n+1}} < \infty,$$

and $\{\beta_n\}$ increasing, as the following theorem shows.

Theorem 2 *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. Fix $u, x_0 \in H$, and let $\{x_n\}$ be the sequence generated by algorithm* (2) *with the conditions:* (i) *and* (ii) *of Theorem 1,* (iii) $\lambda_n, \gamma_n \in [0,1]$, $\alpha_n + \lambda_n + \gamma_n = 1$ *with* $\lim_{n\to\infty} \lambda_n = 0$*, and* (iv) $\beta_n \in (0,\infty)$ *with* $\beta_n \leq \beta_{n+1}$ *for all* $n \geq 0$ *being satisfied. If, in addition, (C2) and (C3) holds, then $\{x_n\}$ converges strongly to $P_F u$, the projection of u on F.*

Proof: According to Lemma 6, $\{x_n\}$ is bounded. Like in the proof of Theorem 1, we have $\omega_{\mathtt{w}}(\{x_n\}) = \omega_{\mathtt{w}}(\{v_n\})$, where $\{v_n\}$ is defined by (3). In order to derive $\omega_{\mathtt{w}}(\{x_n\}) \subset F$, it is enough if we could prove that (6) holds. For this purpose, we compare x_{n+2} and x_{n+1} as follows

$$\begin{aligned} x_{n+2} - x_{n+1} &= \gamma_n (J_{\beta_{n+1}} x_{n+1} - J_{\beta_n} x_n) + \lambda_n (x_{n+1} - x_n) + (\lambda_{n+1} - \lambda_n)(x_{n+1} - J_{\beta_{n+1}} x_{n+1}) \\ &+ (\alpha_{n+1} - \alpha_n)(u - J_{\beta_{n+1}} x_{n+1} + e_{n+1}/\alpha_{n+1}) + \alpha_n (e_{n+1}/\alpha_{n+1} - e_n/\alpha_n). \end{aligned}$$

Using the resolvent identity and the fact that the resolvent is nonexpansive, we get

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\leq \gamma_n \left\| J_{\beta_{n+1}} x_{n+1} - J_{\beta_{n+1}} \left(\tfrac{\beta_{n+1}}{\beta_n} x_n + \left(1-\tfrac{\beta_{n+1}}{\beta_n}\right) J_{\beta_n} x_n \right) \right\| \\
&+ \lambda_n \|x_{n+1}-x_n\| + |\lambda_{n+1}-\lambda_n| K + |\alpha_{n+1}-\alpha_n| L + \alpha_n \left\| \frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n} \right\| \\
&\leq \gamma_n \left\| \tfrac{\beta_{n+1}}{\beta_n} (x_{n+1}-x_n) + \left(1-\tfrac{\beta_{n+1}}{\beta_n}\right)(x_{n+1} - J_{\beta_n} x_n) \right\| \\
&+ \lambda_n \|x_{n+1}-x_n\| + |\lambda_{n+1}-\lambda_n| K + |\alpha_{n+1}-\alpha_n| L + \alpha_n \left\| \frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n} \right\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{\|x_{n+2}-x_{n+1}\|}{\beta_{n+1}} &\leq (1-\alpha_n)\frac{\|x_{n+1}-x_n\|}{\beta_n} + \left(\frac{1}{\beta_n} - \frac{1}{\beta_{n+1}}\right) K + \frac{\alpha_n}{\beta_{n+1}} \left\| \frac{e_{n+1}}{\alpha_{n+1}} - \frac{e_n}{\alpha_n} \right\| \\
&+ \frac{|\lambda_{n+1}-\lambda_n|}{\beta_{n+1}} K + \frac{|\alpha_{n+1}-\alpha_n|}{\beta_{n+1}} L,
\end{aligned}$$

for some constants $K, L > 0$. On the other hand, $\{\beta_n\}$ increasing implies

$$\frac{1}{\beta_n} \text{ is convergent } \Leftrightarrow \sum_{n=0}^{\infty} \left(\frac{1}{\beta_n} - \frac{1}{\beta_{n+1}} \right) < \infty. \tag{9}$$

Therefore, from this fact and Lemma 1, we get (6). The rest of the proof is similar to that of Theorem 1. ■

Theorem 3 *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. Fix $u, x_0 \in H$, and let $\{x_n\}$ be the sequence generated by algorithm* (2) *with the conditions:* (i) *and* (ii) *of Theorem 1,* (iii) $\lambda_n, \gamma_n \in [0,1]$, $\alpha_n + \lambda_n + \gamma_n = 1$ *with* $\lim_{n\to\infty} \lambda_n = 0$, *and* (iv) $\beta_n \in (0,\infty)$ *with* $\lim_{n\to\infty} \beta_n = \infty$, *being satisfied. Then $\{x_n\}$ converges strongly to $P_F u$, the projection of u on F.*

Proof: Observe that for $\beta_n \to \infty$ and $\{x_n\}$ bounded, passing to the limit in (4) immediately yields $\omega_w(\{x_n\}) = \omega_w(\{v_n\}) \subset F$. Again we derive strong convergence of $\{x_n\}$ to $P_F u$ in a similar way as in the proof of Theorem 1. ■

Remark 3: Theorem 3 contains Theorem 1 of [1] as a special case.

We conclude this section by giving a strong convergence result of the sequence $\{x_n\}$ generated by algorithm (2) when $\{\lambda_n\}$ does not converge to zero. The following theorem extends the result of Yao and Noor [17] to general errors.

Theorem 4 *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. Fix $u, x_0 \in H$, and let $\{x_n\}$ be the sequence generated by algorithm* (2) *with the conditions:* (i) *and* (ii) *of Theorem 1,* (iii) $\gamma_n \in (0,1)$, *and* $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$, (iv) $\beta_n \in (0,\infty)$ *with* $\liminf_{n\to\infty} \beta_n > 0$ *and* $\lim_{n\to\infty}(\beta_{n+1} - \beta_n) = 0$, *being satisfied. Then $\{x_n\}$ converges strongly to $P_F u$, the projection of u on F.*

Proof: We note that $\{x_n\}$ is bounded, see Lemma 6. Denoting

$$y_n := \frac{x_{n+1} - \lambda_n x_n}{1-\lambda_n} \quad \text{implies that} \quad y_n = \frac{\alpha_n u + \gamma_n J_{\beta_n} x_n + e_n}{1-\lambda_n}.$$

Therefore, we have

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \left|\frac{\alpha_{n+1}}{1-\lambda_{n+1}} - \frac{\alpha_n}{1-\lambda_n}\right| \|u - J_{\beta_n} x_n\| + \left\|\frac{e_{n+1}}{1-\lambda_{n+1}} - \frac{e_n}{1-\lambda_n}\right\| \\ &+ \frac{\gamma_{n+1}}{1-\lambda_{n+1}} \|(J_{\beta_{n+1}} x_{n+1} - J_{\beta_{n+1}} x_n) + (J_{\beta_{n+1}} x_n - J_{\beta_n} x_n)\| \\ &\leq \left|\frac{\alpha_{n+1}}{1-\lambda_{n+1}} - \frac{\alpha_n}{1-\lambda_n}\right| \|u - J_{\beta_n} x_n\| + \left\|\frac{e_{n+1}}{1-\lambda_{n+1}} - \frac{e_n}{1-\lambda_n}\right\| \\ &+ \frac{\gamma_{n+1}}{1-\lambda_{n+1}} \left(\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \frac{\|x_n - J_{\beta_n} x_n\|}{\beta_n} \right), \end{aligned}$$

where the second inequality follows from the application of the resolvent identity. The last estimate implies that

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0,$$

and therefore by Lemma 5, we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{10}$$

Claim: $\limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u\rangle \leq 0$.

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converging weakly to some x_∞, such that

$$\limsup_{n\to\infty} \langle u - P_F u, x_n - P_F u\rangle = \lim_{k\to\infty} \langle u - P_F u, x_{n_k} - P_F u\rangle = \langle u - P_F u, x_\infty - P_F u\rangle.$$

In order to prove the claim, it suffices to show that $x_\infty \in F$. We note that for any $\beta > 0$,

$$\begin{aligned} 2\langle x_{n_k} - J_\beta x_\infty, x_\infty - J_\beta x_\infty\rangle &= \|x_{n_k} - J_\beta x_\infty\|^2 + \|x_\infty - J_\beta x_\infty\|^2 - \|x_{n_k} - x_\infty\|^2 \\ &\leq (\|x_{n_k} - J_\beta x_{n_k}\| + \|x_{n_k} - x_\infty\|)^2 + \|x_\infty - J_\beta x_\infty\|^2 \\ &- \|x_{n_k} - x_\infty\|^2 \\ &\leq K' \|x_{n_k} - J_\beta x_{n_k}\| + \|x_\infty - J_\beta x_\infty\|^2, \end{aligned}$$

for some $K' > 0$. Hence for fixed $\beta > 0$, we have on passing to the limit in the last inequality as $k \to \infty$

$$\|x_\infty - J_\beta x_\infty\|^2 \leq K' \lim_{k\to\infty} \|x_{n_k} - J_\beta x_{n_k}\|. \tag{11}$$

On the other hand, from (2), we have

$$\begin{aligned} \|x_{n+1} - J_{\beta_n} x_n\| &\leq \alpha_n \|u - J_{\beta_n} x_n + e_n/\alpha_n\| + \lambda_n \|x_n - J_{\beta_n} x_n\| \\ &\leq \alpha_n \|u - J_{\beta_n} x_n + e_n/\alpha_n\| + \lambda_n (\|x_n - x_{n+1}\| + \|x_{n+1} - J_{\beta_n} x_n\|), \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|x_{n+1} - J_{\beta_n}x_n\| = 0. \tag{12}$$

Observe that for any $\beta > 0$, we have by the resolvent identity

$$\begin{aligned}\|J_{\beta_n}x_n - J_\beta x_n\| &\leq \left\|\left(1 - \frac{\beta}{\beta_n}\right)(J_{\beta_n}x_n - x_n)\right\| \\ &\leq \left|1 - \frac{\beta}{\beta_n}\right| (\|J_{\beta_n}x_n - x_{n+1}\| + \|x_{n+1} - x_n\|).\end{aligned}$$

Therefore, for fixed $\beta > 0$, we have from (10) and (12)

$$\lim_{n\to\infty} \|J_{\beta_n}x_n - J_\beta x_n\| = 0. \tag{13}$$

Moreover, from (10), (12) and (13), we get

$$\lim_{n\to\infty} \|x_n - J_\beta x_n\| \leq \lim_{n\to\infty} (\|x_n - x_{n+1}\| + \|x_{n+1} - J_{\beta_n}x_n\| + \|J_{\beta_n}x_n - J_\beta x_n\|) = 0.$$

Combining the last inequality with (11), we have $x_\infty = J_\beta x_\infty$. In other words, $x_\infty \in A^{-1}(0)$, hence the claim is proved.

As before, strong convergence of $\{x_n\}$ to $P_F u$ can be derived. ■

4 A generalized regularization method

In this section, we suggest and analyze a new iterative process for solving problem (1). It is defined as follows: Given any fixed $u, v_1 \in H$, generate a sequence $\{v_n\}$ iteratively by the rule

$$v_{n+1} = J_{\beta_n}(\alpha_{n-1}u + \lambda_{n-1}v_n + \gamma_{n-1}Tv_n + e_{n-1}), \quad \text{for all} \quad n \geq 1, \tag{14}$$

where $T : H \to H$ is a nonexpansive map, $\alpha_n \in (0, 1)$, $\lambda_n, \gamma_n \in [0, 1]$ with $\alpha_n + \lambda_n + \gamma_n = 1$, and $\beta_n \in (0, \infty)$. It is clear that for $\lambda_n = 0$ for all $n \geq 1$, algorithm (14) collapses to

$$v_{n+1} = J_{\beta_n}(\alpha_{n-1}u + (1 - \alpha_{n-1})Tv_n + e_{n-1}), \quad \text{for all} \quad n \geq 1, \tag{15}$$

which is of the same form as the regularization method proposed by Xu [16]. In fact, for $T = I$ (the identity operator), this special case corresponds to the case when $\lambda_n = 0$ for all $n \geq 1$ in algorithm (2). The equivalence of such algorithms was discussed in [2]. Strong convergence of $\{v_n\}$ have been established in [2] under weak assumptions on $\{\alpha_n\}$ and $\{\beta_n\}$ such as

$$(a)\ \sum_{n=1}^{\infty} \left|\frac{\alpha_n}{\beta_{n+1}} - \frac{\alpha_{n-1}}{\beta_n}\right| < \infty, \quad \text{or} \quad (a)'\ \lim_{n\to\infty} \frac{1}{\alpha_n\beta_n^2}(\alpha_n\beta_n - \alpha_{n-1}\beta_{n+1}) = 0.$$

By modifying the arguments contained in [2], we are able to drop the condition that the sequence on $\{\beta_n\}$ is increasing in Theorem 5 [2] and replace (a)$'$ with a slightly better condition

$$(b)\ \lim_{n\to\infty}\frac{1}{\beta_{n+1}}\left(\frac{\alpha_n}{\alpha_{n-1}}-\frac{\beta_{n+1}}{\beta_n}\right)=0,$$

thus we obtain a refinement of Theorem 5 [2], (see Theorem 7 below). In the first theorem of this section, we shall use similar conditions to (a) and (b) on $\{\gamma_n\}$ for the general case when this sequence is not identically zero for all n to derive strong convergence of $\{v_n\}$. These conditions are

$$(c)\ \sum_{n=1}^{\infty}\left|\frac{\gamma_n}{\beta_{n+1}}-\frac{\gamma_{n-1}}{\beta_n}\right|<\infty,\quad\text{and}\quad (d)\ \lim_{n\to\infty}\frac{1}{\alpha_{n-1}}\left(\frac{\gamma_n}{\beta_{n+1}}-\frac{\gamma_{n-1}}{\beta_n}\right)=0.$$

Theorem 5 *Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator and $T : H \to H$ a nonexpansive map with $\emptyset \neq A^{-1}(0) =: F \subset F(T)$, where $F(T)$ is the set of all fixed points of T. For any fixed $u, v_1 \in H$, let the sequence $\{v_n\}$ be generated by algorithm* (14), *where $\alpha_n \in (0,1)$, $\lambda_n, \gamma_n \in [0,1]$ with $\alpha_n + \lambda_n + \gamma_n = 1$, and $\beta_n \in (0,\infty)$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{e_n\}$ satisfy* (i) *$\alpha_n \to 0$ with $\sum_{n=0}^{\infty}\alpha_n=\infty$,* (ii) *either (E1) or (E2),* (iii) *$\liminf_{n\to\infty}\beta_n > 0$, with either (a) or (b) and* (iv) *either (c) or (d). If either $\lambda_n \to 0$ and $\{\beta_n\}$ is bounded, or $\gamma_n \to 0$, then $\{v_n\}$ converges strongly to $P_F u$, the projection of u on F.*

Proof: We divide the proof into three steps. The first step is to show that $\{v_n\}$ is bounded. Assume that $\{\|e_n\|/\alpha_n\}$ is bounded. Then, there exists $M > 0$ such that

$$\|u-p\|+\frac{\|e_n\|}{\alpha_n}\le M \quad \text{for some } p\in F \text{ and all } n\ge 0.$$

Using (14), we see that

$$\begin{aligned}\|v_{n+1}-p\|^2 &\le \|\alpha_{n-1}(u-p+e_{n-1}/\alpha_{n-1})+\lambda_{n-1}(v_n-p)+\gamma_{n-1}(Tv_n-p)\|\\ &\le \alpha_{n-1}\left[\|u-p\|+\frac{\|e_{n-1}\|}{\alpha_{n-1}}\right]+\lambda_{n-1}\|v_n-p\|+\gamma_{n-1}\|Tv_n-p\|\\ &\le \alpha_{n-1}M+(1-\alpha_{n-1})\|v_n-p\|,\end{aligned}$$

and by induction, we have

$$\|v_{n+1}-p\|\le M\left[1-\prod_{k=0}^{n}(1-\alpha_k)\right]+\|v_0-p\|\prod_{k=0}^{n}(1-\alpha_k),$$

showing that $\{v_n\}$ is bounded.

For any $p \in F$ and $\sum_{n=0}^{\infty}\|e_n\|<\infty$, we have

$$\begin{aligned}\|v_{n+1}-p\| &\le \|\alpha_{n-1}(u-p)+\lambda_{n-1}(v_n-p)+\gamma_{n-1}(Tv_n-p)+e_{n-1}\|\\ &\le \alpha_{n-1}\|u-p\|\ +(1-\alpha_{n-1})\|v_n-p\|+\|e_{n-1}\|,\end{aligned}$$

which implies that

$$\|v_{n+1} - p\| \leq \left[1 - \prod_{k=0}^{n}(1-\alpha_k)\right] \|u-p\| + \|v_0 - p\| \prod_{k=0}^{n}(1-\alpha_k) + \sum_{k=0}^{n} \|e_k\|.$$

This shows that $\{v_n\}$ is bounded.

Step 2: The next step is to show that the relation $\omega_{\mathtt{w}}(\{v_n\}) \subset F$ holds. This will suffice to guarantee

$$\limsup_{n\to\infty} \langle u - P_F u, v_n - P_F u\rangle \leq 0.$$

Indeed, for some subsequence of $\{v_n\}$ converging weakly, to some v_∞, we have

$$\limsup_{n\to\infty} \langle u - P_F u, v_n - P_F u\rangle = \lim_{k\to\infty} \langle u - P_F u, v_{n_k} - P_F u\rangle = \langle u - P_F u, v_\infty - P_F u\rangle.$$

For the case when $\|e_n\|/\alpha_n \to 0$, we have, (for some positive constants K and M),

$$\begin{aligned}
\|v_{n+2} - v_{n+1}\| &= \|J_{\beta_{n+1}}(\alpha_n u + \lambda_n v_{n+1} + \gamma_n T v_{n+1} + e_n) \\
&- J_{\beta_{n+1}}\left(\tfrac{\beta_{n+1}}{\beta_n}(\alpha_{n-1}u + \lambda_{n-1}v_n + \gamma_{n-1}Tv_n + e_{n-1}) + (1 - \tfrac{\beta_{n+1}}{\beta_n})v_{n+1}\right)\| \\
&\leq \left\| \tfrac{\gamma_{n-1}\beta_{n+1}}{\beta_n}(Tv_{n+1} - Tv_n) + \tfrac{\lambda_{n-1}\beta_{n+1}}{\beta_n}(v_{n+1} - v_n) \right. \\
&+ \tfrac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\left(\tfrac{e_n}{\alpha_n} - \tfrac{e_{n-1}}{\alpha_{n-1}}\right) + \left(\gamma_n - \tfrac{\gamma_{n-1}\beta_{n+1}}{\beta_n}\right)(Tv_{n+1} - v_{n+1}) \\
&+ \left.\left(\alpha_n - \tfrac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\right)(u - v_{n+1} + \tfrac{e_n}{\alpha_n})\right\| \\
&\leq \tfrac{\beta_{n+1}}{\beta_n}(1-\alpha_{n-1})\|v_{n+1} - v_n\| + \tfrac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\left\|\tfrac{e_n}{\alpha_n} - \tfrac{e_{n-1}}{\alpha_{n-1}}\right\| \\
&+ \left|\gamma_n - \tfrac{\gamma_{n-1}\beta_{n+1}}{\beta_n}\right| K + \left|\alpha_n - \tfrac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\right| M, \qquad (16)
\end{aligned}$$

where equality follows from the resolvent identity and the first inequality from the fact that the resolvent is nonexpansive. Estimate (16) implies that

$$\begin{aligned}
\frac{\|v_{n+2} - v_{n+1}\|}{\beta_{n+1}} &\leq (1-\alpha_{n-1})\frac{\|v_{n+1} - v_n\|}{\beta_n} + \frac{\alpha_{n-1}}{\beta_n}\left\|\frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}}\right\| \\
&+ \left|\frac{\gamma_n}{\beta_{n+1}} - \frac{\gamma_{n-1}}{\beta_n}\right| K + \left|\frac{\alpha_n}{\beta_{n+1}} - \frac{\alpha_{n-1}}{\beta_n}\right| M.
\end{aligned}$$

So if either (a) or (b), and either one of the conditions (c) or (d) is fulfilled, then we have by Lemma 1

$$\frac{\|v_{n+1} - v_n\|}{\beta_n} \to 0. \qquad (17)$$

Note that from (14) we have

$$\frac{(v_{n+1} - v_n)}{\beta_n} + Av_{n+1} \ni \frac{\alpha_{n-1}}{\beta_n}\left(u - v_n + \frac{e_{n-1}}{\alpha_{n-1}}\right) + \frac{\gamma_{n-1}}{\beta_n}(Tv_n - v_n),$$

so that for the case when $\gamma_n \to 0$, we derive $\omega_{\mathtt{w}}(\{v_n\}) \subset F$.

Now we assume that $\lambda_n \to 0$ and $\{\beta_n\}$ is bounded. Then, we have again from (14)

$$\begin{aligned}(v_{n+1} - Tv_{n+1}) + \beta_n A v_{n+1} &\ni \alpha_{n-1}\left(u - Tv_n + e_{n-1}/\alpha_{n-1}\right) + \lambda_{n-1}(v_n - Tv_n) \\ &+ (Tv_n - Tv_{n+1}),\end{aligned}$$

which (together with the monotonicity of A, the boundedness of $\{v_n\}$ and $\{\|e_n\|/\alpha_n\}$, and the fact that T is nonexpansive) implies that

$$\begin{aligned}C(\alpha_{n-1} + \lambda_{n-1} + \|v_n - v_{n+1}\|) &\geq 2\langle v_{n+1} - Tv_{n+1}, v_{n+1} - p\rangle + 2\beta_n\langle Av_{n+1}, v_{n+1} - p\rangle \\ &\geq \|v_{n+1} - Tv_{n+1}\|^2 + \|v_{n+1} - p\|^2 - \|Tv_{n+1} - p\|^2 \\ &\geq \|v_{n+1} - Tv_{n+1}\|^2, \qquad (18)\end{aligned}$$

for some $C > 0$, where p is any point of F. Passing to the limit in (18) and using (17) with $\{\beta_n\}$ bounded, we get

$$\|v_{n+1} - Tv_{n+1}\| \to 0.$$

Moreover, using again (17), we have

$$\begin{aligned}\|v_{n+1} - Tv_n\| &\leq \|v_{n+1} - Tv_{n+1}\| + \|Tv_{n+1} - Tv_n\| \\ &\leq \|v_{n+1} - Tv_{n+1}\| + \|v_{n+1} - v_n\| \to 0. \qquad (19)\end{aligned}$$

On the other hand, from (14), we have

$$(v_{n+1} - Tv_n) + \beta_n A v_{n+1} \ni \alpha_{n-1}(u - Tv_n + e_{n-1}/\alpha_{n-1}) + \lambda_{n-1}(v_n - Tv_n).$$

Since β_n is bounded, this inclusion relation together with (19) imply $\omega_{\mathtt{w}}(\{v_n\}) \subset F$. Note that the proof can be done similarly for the case when (E1) is satisfied.

Step 3: Finally, we establish strong convergence of $\{v_n\}$ to $P_F u$.

Since both T and the resolvent operator are nonexpansive, we have from (14)

$$\begin{aligned}\|v_{n+1} - P_F u\|^2 &\leq \|\alpha_{n-1}(u - P_F u + e_{n-1}/\alpha_{n-1}) + \lambda_{n-1}(v_n - P_F u) + \gamma_{n-1}(Tv_n - P_F u)\|^2 \\ &\leq (1 - \alpha_{n-1})\|v_n - P_F u\|^2 + \alpha_{n-1}^2\|u - P_F u + e_{n-1}/\alpha_{n-1}\|^2 \\ &+ 2\alpha_{n-1}\langle u - P_F u + e_{n-1}/\alpha_{n-1}, \lambda_{n-1}(v_n - P_F u) + \gamma_{n-1}(Tv_n - P_F u)\rangle.\end{aligned}$$

Therefore if either $\gamma_n \to 0$ or $\lambda_n \to 0$ and $\{\beta_n\}$ is bounded, then by Lemma 1, we derive $v_n \to P_F u$. The proof is similar for the case when (E1) is satisfied. ■

Theorem 6 *Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator and $T : H \to H$ a nonexpansive map with $\emptyset \neq A^{-1}(0) =: F \subset F(T)$, where $F(T)$ is the set of all fixed points of T. For any fixed $u, v_1 \in H$, let the sequence $\{v_n\}$ be generated by algorithm* (15)*, where $\alpha_n \in (0,1)$ and $\beta_n \in (0,\infty)$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{e_n\}$ satisfy* (i) *$\alpha_n \to 0$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$,* (ii) *either (E1) or (E2),* (iii) *$\liminf_{n\to\infty} \beta_n > 0$, with (e) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ and either (a) or*

$$(b)' \lim_{n\to\infty}\left(\frac{\alpha_n}{\alpha_{n-1}} - \frac{\beta_{n+1}}{\beta_n}\right) = 0.$$

Then $\{v_n\}$ converges strongly to $P_F u$, the projection of u on F.

Proof: We know from the proof of Theorem 5 that $\{v_n\}$ is bounded. For $\|e_n\|/\alpha_n \to 0$, we have, (by the resolvent identity and the nonexpansivity of the resolvent),

$$\begin{aligned}
\|v_{n+2}-v_{n+1}\| &= \|J_{\beta_{n+1}}(\alpha_n u+(1-\alpha_n)Tv_{n+1}+e_n) \\
&- J_{\beta_{n+1}}\left(\tfrac{\beta_{n+1}}{\beta_n}((1-\alpha_{n-1})Tv_n+\alpha_{n-1}u+e_{n-1})+(1-\tfrac{\beta_{n+1}}{\beta_n})v_{n+1}\right)\| \\
&\leq \Big\|\tfrac{\beta_{n+1}}{\beta_n}(1-\alpha_{n-1})(Tv_{n+1}-Tv_n)+\left(1-\tfrac{\beta_{n+1}}{\beta_n}\right)(Tv_{n+1}-v_{n+1}) \\
&+ \left(\alpha_n-\tfrac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\right)(u-Tv_{n+1}+\tfrac{e_n}{\alpha_n})+\tfrac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\left(\tfrac{e_n}{\alpha_n}-\tfrac{e_{n-1}}{\alpha_{n-1}}\right)\Big\| \\
&\leq \tfrac{\beta_{n+1}}{\beta_n}(1-\alpha_{n-1})\|v_{n+1}-v_n\|+\left|1-\tfrac{\beta_{n+1}}{\beta_n}\right|\|Tv_{n+1}-v_{n+1}\| \\
&+ \left|\alpha_n-\tfrac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\right|\|u-Tv_{n+1}+\tfrac{e_n}{\alpha_n}\|+\tfrac{\alpha_{n-1}\beta_{n+1}}{\beta_n}\left\|\tfrac{e_n}{\alpha_n}-\tfrac{e_{n-1}}{\alpha_{n-1}}\right\|, \qquad (20)
\end{aligned}$$

so that for some positive constants K and M, we have

$$\begin{aligned}
\frac{\|v_{n+2}-v_{n+1}\|}{\beta_{n+1}} &\leq (1-\alpha_{n-1})\frac{\|v_{n+1}-v_n\|}{\beta_n}+\left|\frac{1}{\beta_{n+1}}-\frac{1}{\beta_n}\right|K+\left|\frac{\alpha_n}{\beta_{n+1}}-\frac{\alpha_{n-1}}{\beta_n}\right|M \\
&+ \frac{\alpha_{n-1}}{\beta_n}\left\|\frac{e_n}{\alpha_n}-\frac{e_{n-1}}{\alpha_{n-1}}\right\|.
\end{aligned}$$

So, if condition (e) and either (a) or (b)$'$ are fulfilled, then we have by Lemma 1

$$\frac{\|v_{n+1}-v_n\|}{\beta_n}\to 0 \quad \Leftrightarrow \quad \|v_{n+1}-v_n\|\to 0. \qquad (21)$$

Now proceeding in a similar way as in the proof of Theorem 5 (with $\lambda_n = 0$ for all n), we derive strong convergence of $\{v_n\}$ to $P_F u$. The proof is similar for the case when (E1) is satisfied ■

Remark 4: Note that conditions (a)$'$ and (b)$'$ are not comparable in general. For example

$$\beta_{n+1} = n\beta_n, \quad \text{and} \quad \alpha_n = \frac{1}{n}+(-1)^n\frac{1}{n+1}$$

satisfy (a)$'$ but not (b)$'$. Of course both of them are satisfied for $\alpha_n = n^{-1}$ and $\beta_n = n$.

Remark 5: We observe that the condition that β_n be bounded in Theorems 5 and 6 is superfluous if T is linear. Of course this is the case also when H is finite dimensional. In these two cases, Theorem 6 holds under the more general condition

$$\sum_{n=0}^{\infty}\left|\frac{1}{\beta_{n+1}}-\frac{1}{\beta_n}\right|<\infty$$

instead of condition (e). In fact, it still holds even for any $\beta_n \to \infty$ as shown in the following corollary. Such conditions on β_n are not comparable in general. Indeed, for any $\{\beta_n\}$ bounded below away from zero, one can check that

$$\sum_{n=0}^{\infty}\left|\frac{1}{\beta_{n+1}}-\frac{1}{\beta_n}\right| = \infty \quad \text{for} \quad \beta_n = \begin{cases} 2n, & \text{if } n \text{ is odd,} \\ n+1, & \text{if } n \text{ is even.}\end{cases}$$

However, for increasing $\{\beta_n\}$, the condition (9) is implied by $\beta_n \to \infty$.

Corollary 1 *Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator and $T : H \to H$ a nonexpansive map with $\emptyset \neq A^{-1}(0) =: F \subset F(T)$, where $F(T)$ is the set of all fixed points of T. For any fixed $u, v_1 \in H$, let the sequence $\{v_n\}$ be generated by algorithm* (15), *where $\alpha_n \in (0,1)$ and $\beta_n \in (0,\infty)$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{e_n\}$ satisfy conditions* (i) *and* (ii) *of Theorem 6, and* (iii) *$\beta_n \to \infty$. If T is linear, then $\{v_n\}$ converges strongly to $P_F u$, the projection of u on F.*

Proof: We note that from (15), we have

$$\frac{(v_{n+1} - Tv_n)}{\beta_n} + Av_{n+1} \ni \frac{\alpha_{n-1}}{\beta_n}(u - Tv_n) + \frac{e_{n-1}}{\beta_n}.$$

Since $\beta_n \to \infty$, this inclusion relation gives $\omega_{\mathtt{w}}(\{v_n\}) \subset F$. Since T is linear, $\omega_{\mathtt{w}}(\{v_n\}) = \omega_{\mathtt{w}}(\{Tv_n\})$. The rest of the proof is similar to that of Theorem 6. ■

Theorem 7 *Let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator with $\emptyset \neq A^{-1}(0) =: F$. For any fixed $u, v_1 \in H$, let the sequence $\{v_n\}$ be generated by algorithm* (15) *with $T = I$ (the identity operator), where $\alpha_n \in (0,1)$ and $\beta_n \in (0,\infty)$. Assume that $\{\alpha_n\}$, and $\{e_n\}$ satisfy conditions* (i) *and* (ii) *of Theorem 6. If $\liminf_{n\to\infty} \beta_n > 0$, and either one of the conditions (a) or (b) are met, then $\{v_n\}$ converges strongly to $P_F u$, the projection of u on F.*

Proof: Assume $\|e_n\|/\alpha_n \to 0$. We know that $\{v_n\}$ is bounded, see Theorem 5.

Claim: $\limsup_{n\to\infty} \langle u - P_F u, v_n - P_F u\rangle \leq 0$.

Let $\{v_{n_k}\}$ be a subsequence of $\{v_n\}$ converging weakly, to some v_∞, such that

$$\limsup_{n\to\infty} \langle u - P_F u, v_n - P_F u\rangle = \lim_{k\to\infty} \langle u - P_F u, v_{n_k} - P_F u\rangle = \langle u - P_F u, v_\infty - P_F u\rangle.$$

To prove the claim, we only need to show that $v_\infty \in F$, or more generally $\omega_{\mathtt{w}}(\{v_n\}) \subset F$. From equation (20) with $T = I$, we have for some $M > 0$

$$\frac{\|v_{n+2} - v_{n+1}\|}{\beta_{n+1}} \leq (1 - \alpha_{n-1}) \frac{\|v_{n+1} - v_n\|}{\beta_n} + \left| \frac{\alpha_n}{\beta_{n+1}} - \frac{\alpha_{n-1}}{\beta_n} \right| M + \frac{\alpha_{n-1}}{\beta_n} \left\| \frac{e_n}{\alpha_n} - \frac{e_{n-1}}{\alpha_{n-1}} \right\|,$$

so that if either one of the conditions (a) or (b) is fulfilled, then we establish

$$\frac{\|v_{n+1} - v_n\|}{\beta_n} \to 0.$$

Moreover, we get from (15) with $T = I$

$$\frac{v_{n+1} - v_n}{\beta_n} + A(v_{n+1}) \ni \frac{\alpha_{n-1}}{\beta_n} (u - v_n) + \frac{1}{\beta_n} e_{n-1},$$

which implies that $\omega_{\mathtt{w}}(\{v_n\}) \subset F$, hence the claim.

Again proceeding in a similar way as in the proof of Theorem 5 (with $\gamma_n = 0$ for all n), we derive strong convergence of $\{v_n\}$ to $P_F u$. ■

4.1 The case when A is a subdifferential

We remark that for $A = \partial\varphi$ − the subdifferential of a proper, convex and lower semicontinuous function $\varphi : H \to (-\infty, +\infty]$, algorithm (14) is equivalent to

$$v_{n+1} = \arg\min_{x\in H}\varphi_n(x),$$

where

$$\varphi_n(x) = \varphi(x) + \frac{1}{2\beta_n}\|x - \alpha_{n-1}u - \lambda_{n-1}v_n - \gamma_{n-1}Tv_n - e_{n-1}\|^2.$$

The function φ_n is somewhat favorable compared to the original function φ as it preserve all the properties of φ, and even more, it is always coercive having a unique minimizer v_{n+1} due to the quadratic term added to φ. Note that under the assumptions of Theorems 5, 6 and 7, $\{v_n\}$ converges strongly to the minimizer of φ nearest to u. In fact, in this case of the subdifferential, we can say a little bit more, namely, that under fairly mild conditions, we can show that $\varphi(w_k)$ converges to the infimum of φ, where

$$w_n = \sigma_n^{-1}\sum_{k=1}^{n}\beta_k v_{k+1}, \quad \text{with} \quad \sigma_n = \sum_{k=1}^{n}\beta_k. \tag{22}$$

Theorem 8 *Let $A = \partial\varphi$ and $\emptyset \neq A^{-1}(0) \subset F(T)$ where $\varphi : H \to (-\infty, +\infty]$ is a proper, convex and lower semicontinuous function, and $F(T)$ is the set of all fixed points of the nonexpansive map $T : H \to H$. For any fixed $u, v_1 \in H$, let $\{v_n\}$ be the sequence generated by algorithm (14) and $\{w_n\}$ be as in (22).*

- *If $\{e_n/\alpha_n\}$ is bounded, then for some $M > 0$, we have*

$$\varphi(w_n) - \varphi(z) \leq \frac{\|v_1 - z\|^2 + M\sum_{k=0}^{n-1}(\alpha_k + \gamma_k)}{2\sigma_n}, \quad \textit{for all} \quad z \in H. \tag{23}$$

- *If $\sum_{k=0}^{\infty}\|e_k\| < \infty$, then for some $K > 0$, the following estimate holds*

$$\varphi(w_n) - \varphi(z) \leq \frac{\|v_1 - z\|^2 + K(\sum_{k=0}^{n-1}\alpha_k + \gamma_k + \|e_k\|)}{2\sigma_n}, \quad \textit{for all} \quad z \in H. \tag{24}$$

If in addition, $\sigma_n^{-1}\sum_{k=0}^{n-1}(\alpha_k + \gamma_k) \to 0$ as $n \to \infty$, then $\varphi(w_n) \to \inf_{y\in H}\varphi(y)$.

Proof: We prove only estimate (23). The proof of the other estimate is similar. Note that for $A = \partial\varphi$, we have from (14),

$$\alpha_{k-1}(u - v_k + e_{k-1}/\alpha_{k-1}) + \gamma_{k-1}(Tv_k - v_k) + (v_k - v_{k+1}) \in \beta_k\partial\varphi(v_{k+1}),$$

and since both $\{e_k/\alpha_k\}$ and $\{v_k\}$ are bounded, we have for all $z \in H$

$$\begin{aligned}
2\beta_k(\varphi(v_{k+1}) - \varphi(z)) \;&\leq\; 2\langle v_k - v_{k+1}, v_{k+1} - z\rangle + \gamma_{k-1}\langle Tv_k - v_k, v_{k+1} - z\rangle \\
&+\; 2\alpha_{k-1}\langle u - v_k + e_{k-1}/\alpha_{k-1}, v_{k+1} - z\rangle \\
&\leq\; (\|v_k - z\|^2 - \|v_{k+1} - v_k\|^2 - \|v_{k+1} - z\|^2) \\
&+\; M(\gamma_{k-1} + \alpha_{k-1}).
\end{aligned} \tag{25}$$

for some $M > 0$. Summing (25) from $k = 1, \ldots, n$ and rearranging terms, we get

$$2\varphi(z) + \frac{\|v_1 - z\|^2 + M\sum_{k=1}^{n}(1-\lambda_{k-1})}{\sigma_n} \geq 2\frac{\sum_{k=1}^{n}\beta_k\varphi(v_{k+1})}{\sigma_n} \geq 2\varphi\left(\frac{\sum_{k=1}^{n}\beta_k v_{k+1}}{\sigma_n}\right). \quad (26)$$

Therefore (23) follows from (26). The final assertion of the theorem is obvious. ■

Theorem 9 *Let $A = \partial\varphi$ and $\emptyset \neq A^{-1}(0) \subset F(T)$ where $\varphi : H \to (-\infty, +\infty]$ is a proper, convex and lower semicontinuous function, and $F(T)$ is the set of all fixed points of the nonexpansive map $T : H \to H$. For any fixed $u, v_1 \in H$, let $\{v_n\}$ be the sequence generated by algorithm (14) with $\liminf_{n\to\infty}\beta_n > 0$ and $\sum_{k=0}^{\infty}(\alpha_k + \gamma_k)\beta_{k+1}^{-1} < \infty$ being satisfied. If either $\{\|e_n\|/\alpha_n\}$ is bounded, or $\sum_{k=0}^{\infty}\|e_k\| < \infty$, then $\varphi(v_n) \to \inf_{y\in H}\varphi(y)$ as $n \to \infty$. Moreover, if either φ has a unique minimizer, or $\{\beta_n\}$ is bounded, then $\{v_n\}$ converges weakly to the minimizer of φ.*

Proof: We prove the result only for the case when $\{\|e_n\|/\alpha_n\}$ is bounded. The proof of the other case is similar. Note from (25) (with $z := v_n$), we have

$$\varphi(v_{n+1}) - \frac{M}{2}\sum_{k=0}^{n-1}\frac{(\alpha_k + \gamma_k)}{\beta_{k+1}} \leq \varphi(v_n) - \frac{M}{2}\sum_{k=0}^{n-2}\frac{(\alpha_k + \gamma_k)}{\beta_{k+1}},$$

and since

$$\varphi(v_{n+1}) - \frac{M}{2}\sum_{k=0}^{n-1}\frac{(\alpha_k + \gamma_k)}{\beta_{k+1}} \geq \inf_{y\in H}\varphi(y) - \frac{M}{2}\sum_{k=0}^{\infty}\frac{(\alpha_k + \gamma_k)}{\beta_{k+1}} > -\infty,$$

we conclude that $\lim_{n\to\infty}\varphi(v_n)$ exists and is finite. Again we note from (25), (with $z := v_n$), that

$$\left(\frac{\|v_{n+1} - v_n\|}{\beta_n}\right)^2 \leq \frac{1}{\beta_n}\left[\varphi(v_n) - \varphi(v_{n+1}) + \frac{M(\alpha_{n-1} + \gamma_{n-1})}{2\beta_n}\right] \to 0. \quad (27)$$

Moreover, for any $z \in H$, we have from (25)

$$\varphi(v_{n+1}) - \varphi(z) \quad \leq \quad \left\langle \frac{v_n - v_{n+1}}{\beta_n}, v_{k+1} - z\right\rangle + \frac{M(\alpha_{n-1} + \gamma_{n-1})}{2\beta_n} \to 0.$$

In particular,

$$\lim_{n\to\infty}\varphi(v_n) = \inf_{y\in H}\varphi(y). \quad (28)$$

We have from (14),

$$\partial\varphi(v_{k+1}) \ni \frac{\alpha_{k-1}}{\beta_k}(u - v_k + \frac{e_{k-1}}{\alpha_{k-1}}) + \frac{\gamma_{k-1}}{\beta_k}(Tv_k - v_k) + \frac{(v_k - v_{k+1})}{\beta_k} \to 0.$$

Since $\partial\varphi$ is demiclosed, we conclude that $v_\infty \in F$. Thus $\omega_{\mathtt{w}}(\{v_n\}) \subset F$. If φ has a unique minimizer, i.e., F is a singleton, the conclusion follows. Now assume that $\{\beta_n\}$ is bounded, and let $\{v_{n_k}\}$ be a subsequence of $\{v_n\}$ converging weakly to some v_∞. Then, for any $p \in F$, we have from (25)

$$\|v_{n+1} - p\|^2 - M\sum_{k=0}^{n-1}(\alpha_k + \gamma_k) \quad \leq \quad \|v_n - p\|^2 - M\sum_{k=0}^{n-2}(\alpha_k + \gamma_k).$$

On the other hand,

$$\|v_{n+1} - p\|^2 - M\sum_{k=0}^{n-1}(\alpha_k + \gamma_k) \geq -M\sum_{k=0}^{\infty}(\alpha_k + \gamma_k) > -\infty.$$

These two inequalities imply that $\lim_{n\to\infty}\|v_n - p\|$ exists and it is finite. Therefore, by Opial's lemma, there exists $q \in F$ such that $v_n \rightharpoonup q$. ∎

5 Concluding remarks

Two distinct four parameter proximal point algorithms were considered and strong convergence results associated with them were proved under different sets of conditions on these parameters. Loosely speaking, the results presented hold true for any sequence of errors $\{e_n\}$ converging to zero in norm. For such a sequence of error terms, one constructs a sequence of parameters $\{\alpha_n\} \subset (0,1)$ in such a way that the condition $\|e_n\|/\alpha_n \to 0$ is fulfilled. Perhaps an interesting case is when such a sequence of errors satisfy the condition $\sum_{n=0}^{\infty}\|e_n\| = \infty$, in which case $\sum_{n=0}^{\infty}\alpha_n = \infty$ follows automatically. In that case we may for instance choose $\alpha_n = \sqrt{\|e_n\|}$ if $e_n \neq 0$ and n is large enough, and $\alpha_n = 1/(n+2)$ otherwise. Obviously, such a sequence of parameters $\{\alpha_n\}$ belongs to the interval $(0,1)$ with $\alpha_n \to 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, and condition (E2) is fulfilled. Note that the resulting parameters, α_n's, constructed in this way depend on $\{e_n\}$, but this is acceptable as far as numerical analysis is concerned. It is worth pointing out that, one may in fact assume that the sequence $\{e_n\}$ is of the form $e_n = e'_n + e^*_n$, where $\{e'_n\}$ satisfy condition (E1) and $\{e^*_n\}$ satisfy (E2). Such a reformulation facilitates a neat and unified discussion on errors.

In the case when $A = \partial\varphi$, it should be noted that for each $n \geq 1$, v_n as defined by algorithm (14) always belong to the domain of φ. This is a favorable feature (which is not provided for in the other algorithm) which allows us to approximate minimum points of the functional φ. However, this property is shared by both algorithms in the case when λ_n is identically zero for all n and T is the identity operator, since for this case the two algorithms are equivalent. Concerning the sequence of errors (when they are not summable), we emphasize that for the case of the subdifferential, we were able to increase the pool of items from which the sequence of parameters $\{\alpha_n\}$ can be chosen in order to construct sequences that approximate minimum values of the convex functional φ. A typical construction of such a sequence can be obtained by choosing the parameters α_n's in such a way that $\alpha_n = \|e_n\|$ for $e_n \neq 0$ if n stays large, and $\alpha_n = 1/2n$ otherwise.

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