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On long-term arbitrage opportunities in Markovian models of financial markets

Martin L. D. Mbele Bidima · Miklós Rásonyi

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Abstract A discrete-time infinite horizon stock market model is considered where the logarithm of the price is assumed to be a Markov chain arising from the time-discretization of a stochastic differential equation.

Conditions are given which ensure that there exist investment strategies producing an exponential growth of wealth with a probability converging to 1. The rate of this convergence is studied using large deviation techniques.

Keywords asymptotic arbitrage · large deviations · Markov chains · loss probability

1 Introduction

Most models of mathematical finance in continuous or discrete time share the following feature: for each finite time horizon $T > 0$, there are no arbitrage opportunities to be realized but when $T \to \infty$ is considered one may generate riskless profit in the limit.

Such long-term arbitrage opportunities were studied in a number of papers, see e.g. [2], [3], [4] and [6]. Our starting point is [5] and we intend to settle certain issues raised there.

In [5] the authors considered a d-dimensional diffusion process

$$dS_t = \Sigma(S_t)(dW_t + \phi(S_t)dt), \tag{1}$$

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Martin L. D. Mbele Bidima
Department of Mathematics and its Applications,
Central European University, Budapest.
E-mail: mbele-bidima@ceu-budapest.edu

Miklos Rasonyi, corresponding author.
School of Mathematics, University of Edinburgh
JCMB King's Buildings, Mayfield Road
Edinburgh, EH9 3JZ, UK
Tel.: +44-131-6517677, Fax: +44-131-6506553
E-mail: Miklos.Rasonyi@ed.ac.uk
On leave from MTA SZTAKI, Budapest.

where $W_t, t \geq 0$ is an N-dimensional standard Brownian motion. S_t is thought to represent the price evolution of d risky assets such as stocks, $\Sigma : \mathbb{R}^d \to \mathbb{R}^{d\times N}$ is the volatility matrix that determines the correlations between the assets, $\phi : \mathbb{R}^d \to \mathbb{R}^N$ is the so-called market price of risk function.

The latter has a straightforward interpretation when $d = N = 1$: it is the stock's rate of return per unit volatility. In other words, the drift $\Sigma\phi$ represesents the rate(s) of return on the stock(s).

In [5] a technical condition is assumed: the existence of the so-called minimal martingale measure. We do not elaborate on this point as it is irrelevant to the main line of the discussions below.

Definition 1 (Definition 1.3 of [5].) We say that S has a non-trivial market price of risk if there is $c > 0$ such that

$$\lim_{T\to\infty} P\left(\frac{1}{T}\int_0^T |\phi(S_t)|^2 dt < c\right) = 0. \tag{2}$$

Intuitively, in such a market even if $T \to \infty$ there is still a nonvanishing "drift normalized by the volatility" (i.e. $\phi(S_t)$) which, in some sense, means that market opportunities do not "run dry" as time goes on. As pointed out in [5], (2) holds whenever S_t is "ergodic" (it satisfies a suitable law of large numbers) and has an invariant measure τ such that ϕ is non-zero τ-a.s.

A reformulation of Theorem 1.4 of [5] is stated next.

Theorem 1 *If S has a nontrivial market price of risk then there exists $\gamma > 0$ and for each $\varepsilon > 0$ there exists T_ε such that for all $T > T_\varepsilon$*

$$P(X_T \geq e^{\gamma T}) \geq 1 - \varepsilon$$

for some $X_T \geq -e^{-\gamma T}$, where X_T is the outcome of an admissible trading strategy on $[0, T]$ starting from 0 initial capital.

Since this result serves only as a motivation for our work we do not provide a definition of admissible trading strategies here but rather underline the essential content of Theorem 1: it says that for any tolerance level ε one may find T large enough such that an exponentially growing profit can be obtained on $[0, T]$ with an exponentially decreasing potential loss and with a probability of failure below ε. This can be considered as a rather strong form of long-term arbitrage.

Nevertheless, there are unsettling features of Theorem 1: the relationship between ε and T_ε is not clarified (one may need to wait for a very long time to achieve a desired tolerance level) and the trading strategies are not explicitly given (indeed, the proof is non-constructive). The authors of [5] formulated another condition on the market price of risk that is stronger than (2).

Definition 2 (Definition 1.3 in [5].) The market price of risk satisfies a large deviation estimate if there are $c_1, c_2 > 0$ such that

$$\limsup_{T\to\infty} \frac{1}{T} \ln P\left(\frac{1}{T}\int_0^T |\phi(S_t)|^2 dt \leq c_1\right) \leq -c_2. \tag{3}$$

It is an appealing conjecture that whenever (3) holds, a strengthening of Theorem 1 should hold true: the existence of $\gamma_1, \gamma_2, C > 0$ such that

$$P(X_T \le e^{\gamma_1 T}) \le Ce^{-\gamma_2 T} \tag{4}$$

for all T large enough and for suitable outcomes $X_T \ge -e^{-\gamma_1 T}$ of admissible trading strategies. Such a result would establish an explicit relationship between a preset tolerance level ε and the time necessary to reach that tolerance level. In particular, the probability of having a loss (i.e. $P(X_T < 0)$) could be controlled.

At the end of section 3 in [5] the authors sketch how large deviations theory could be applied to show (4) from (3), but they do not carry out this programme: it seems that some of the necessary theoretical tools are still lacking. They only study the concrete case of the Ornstein-Uhlenbeck process, where explicit γ_1, γ_2 are given, together with an array of convergence rates for related optimization problems.

The main contribution of the present paper is to prove, in a discrete-time version of the model (1), the implication (3)⇒(4) and to provide easily verifiable conditions that guarantee (3) (see section 3). The strategies we use will be explicit but will depend on the particular discretization of (1), so we prove new versions of Theorem 1, too (see Theorem 2 and Corollary 1 in section 2).

2 Asymptotic exponential arbitrage

For simplicity, we consider only one stock whose (positive) price process will evolve according to a time-discretized version of (1). At the same time, we make some natural restrictions on trading that are absent in [5]: neither short selling of the risky asset nor borrowing money will be permitted. There is a credit line requirement in [5] stipulating that admissible strategies must have a value process bounded from below by a constant. Instead of this we will draw the natural credit line 0: investors' wealth processes will not be allowed to go below 0. From now on we switch to a rigorous presentation.

Let $(\Omega, \mathcal{F}, P)$ be a probability space in which all the objects defined in this article will live. E will denote expectation with respect to P. We assume that $X_t, t \ge 0$ is a discrete-time $\mathbb{R}$-valued Markov process such that $X_0 \in \mathbb{R}$ is a constant and for $t \ge 0$,

$$X_{t+1} - X_t = \mu(X_t) + \sigma(X_t)\varepsilon_{t+1}, \tag{5}$$

where $\varepsilon_i, i \ge 1$ is an $\mathbb{R}$-valued i.i.d. sequence of random variables (representing the random driving process of the price evolution), $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are measurable functions. Denote $\mathcal{F}_t = \sigma(X_0, \dots, X_t)$ for $t \ge 0$, the natural filtration of X_t.

We assume that the price of the risky asset (stock) is given by $S_t := \exp\{X_t\},\ t \ge 0$ and that there is a riskless asset whose price equals 1 at all times.

Trading strategies are $(\mathcal{F}_t)_{t \ge 0}$-predictable $[0,1]$-valued processes $\pi_t, t \ge 1$ (that is, π_t is $\mathcal{F}_{t-1}$-measurable). π_t represents the proportion of wealth allocated to the risky asset at time t. (This has to be chosen before the price S_t is revealed, that's why predictability is imposed on the strategy.)

The value process of trading strategy $\pi_t, t \ge 1$, starting from initial capital $V_0^\pi = V_0 > 0$ is defined via

$$V_{t+1}^\pi = V_t^\pi \left[1 - \pi_{t+1} + \pi_{t+1}\frac{S_{t+1}}{S_t}\right] \text{ for } t \ge 0,$$

where we do not signal dependence on V_0 for the sake of a lighter notation.

Definition 3 We say that there is *asymptotic exponential arbitrage* (AEA) if there exist $\gamma > 0$ and a trading strategy $\pi_t,\ t \geq 0$ such that, for all $\varepsilon > 0$ there is $T_\varepsilon \in \mathbb{N}$ satisfying

$$P(V_T^\pi \geq e^{\gamma T}) \geq 1 - \varepsilon,\ \text{for all } T \geq T_\varepsilon. \tag{6}$$

We point out that Definition 3 is seemingly quite different from the conclusion of Theorem 1. AEA is about the existence of a single trading strategy π_t producing arbitrage as $t \to \infty$ while Theorem 1 (translated into our setting) gives for each ε and $T > T_\varepsilon$ (possibly different) $\pi_t(\varepsilon, T)$ satisfying *both* (6) and a geometrically decreasing (in T) loss bound on V_T^π. It turns out, however, that AEA implies this latter kind of arbitrage, too.

Proposition 1 *If there is AEA then for each $\varepsilon > 0$ there exist T_ε and trading strategies $\pi_t(\varepsilon, T), t \geq 1,\ T \geq T_\varepsilon$ such that $V_T^{\pi(\varepsilon,T)} \geq V_0 - e^{-\gamma T/2}$ and*

$$P(V_T^{\pi(\varepsilon,T)} \geq e^{\gamma T/2}) \geq 1 - \varepsilon,\ \textit{for all } T \geq T_\varepsilon. \tag{7}$$

Proof We may and will assume $V_0 = 1$ for the portfolio realizing AEA as well as for the portfolio we are about to construct. Fix $\varepsilon > 0$, take π_t and T_ε as in Definition 3, fix also $T \geq T_\varepsilon$ and define recursively

$$\tilde{\pi}_t := \frac{V_{t-1}^\pi e^{-\gamma T/2} \pi_t}{V_{t-1}^{\tilde{\pi}}}, \quad t \geq 1.$$

One can check that $V_T^{\tilde{\pi}} = V_T^\pi e^{-\gamma T/2} + 1 - e^{-\gamma T/2}$, hence $V_T^{\tilde{\pi}} \geq 1 - e^{-\gamma T/2}$ indeed holds and we also have

$$P(V_T^{\tilde{\pi}} \geq e^{\gamma T/2}) \geq 1 - \varepsilon, \tag{8}$$

showing (7) for $\pi(\varepsilon, T) := \tilde{\pi}$. □

Definition 4 We say that there is *asymptotic exponential arbitrage with geometrically decaying failure probabilities* if there exist $C, \kappa, \gamma > 0$ and a trading strategy $\pi_t,\ t \geq 0$ such that

$$P(V_t^\pi \leq e^{\kappa t}) \leq C e^{-\gamma t},\ \text{for all } t \geq 1.$$

One of the key quantities in our analysis of asymptotic arbitrage is

$$r(x) := E[\exp\{\mu(x) + \sigma(x)\varepsilon_1\}] - 1.$$

This function will play the role of ϕ (see (1)) in the present setting. It has a clear economic interpretation:

$$E[\exp\{\mu(x) + \sigma(x)\varepsilon_1\}] = E\left[\frac{S_{t+1}}{S_t}\,|X_t = x\right],$$

i.e. when the current log-price of the stock is x, the expected value of the return on one unit of stock equals $E[\exp\{\mu(x) + \sigma(x)\varepsilon_1\}]$. Let us recall that the price of the riskless asset is assumed to be 1 all the time and thus has constant 1 expected rate of return. Hence $r(x)$ shows how much the (one-step) future return on one unit of stock exceeds the return on one unit of bond when the current log-price equals x.

Remark 1 A very naive investment strategy is buying stock at time t only if $r(X_t) > 0$, i.e. the expected return of the stock is better than that of the bond. We will see that this simple idea produces a trading strategy yielding AEA under appropriate conditions.

We now state and prove a version of Theorem 1 in the present context.

Theorem 2 *Assume that μ, σ are bounded,*

$$Ee^{a|\varepsilon_1|} < \infty \tag{9}$$

for all $a > 0$ and for some $c > 0$,

$$\lim_{T\to\infty} P\left(\frac{1}{T}\sum_{i=1}^{T} r^2(X_{i-1})1_{\{r(X_{i-1})>0\}} < c\right) = 0. \tag{10}$$

Then there is AEA.

Remark 2 (10) is the analogue of (2) in our setting. As we have already mentioned, r corresponds to ϕ, this will become more clear from Example 2 and Theorem 3 below, see also the discussion at the end of this section.

A conspicuous feature of (10) is the presence of the indicator $1_{\{r(X_i)>0\}}$. This is not surprising: [5] allows short-selling of the stock and hence a negative $r(X_i)$ is just as useful for creating arbitrage as a positive one. Under our short-selling constraint, however, only positive $r(X_i)$ may promise future profit.

We embark on the proof of Theorem 2, and assume from now on that all its hypotheses hold. First let us define the random functions

$$L_x(\pi) := \ln(1 - \pi + \pi \exp\{\mu(x) + \sigma(x)\varepsilon_1\}), \quad \pi \in [0, 1],$$

for all $x \in \mathbb{R}$.

We can see that

$$L'_x(\pi) = \frac{-1 + \exp\{\mu(x) + \sigma(x)\varepsilon_1\}}{1 - \pi + \pi \exp\{\mu(x) + \sigma(x)\varepsilon_1\}}, \tag{11}$$

$$L''_x(\pi) = \frac{-(\exp\{\mu(x) + \sigma(x)\varepsilon_1\} - 1)^2}{(1 - \pi + \pi \exp\{\mu(x) + \sigma(x)\varepsilon_1\})^2} \tag{12}$$

where differentiation is w.r.t. π and the derivatives exist for $\pi \in (0, 1)$ and these functions can be continuously extended to $[0, 1]$.

We need the following estimates.

Proposition 2 *The function $u_x(\pi) := EL_x(\pi)$ is well-defined and*

$$|u_x(\pi)| \le C$$

for some $C > 0$, independent of x and π.

Proof We obviously have

$$1 - \pi(x) + \pi(x)\exp\{\mu(x) + \sigma(x)\varepsilon_1\} \le 1 + \exp\{K(1 + |\varepsilon_1|)\}, \qquad (13)$$

where K is a common bound for $|\mu|, |\sigma|$. For all $0 \le \varpi \le 1/2$ and $y \in \mathbb{R}$ we have

$$1 - \varpi + \varpi \exp\{y\} \ge 1/2$$

and for $1 \ge \varpi > 1/2$ it is $\ge (1/2)\exp\{y\}$. Thus

$$L_x(\pi) \ge \min\{\ln 1/2, \ln(1/2) - K(1 + |\varepsilon_1|)\}. \qquad (14)$$

The statement follows from (13) and (14) as ε_1 is integrable (even exponentially integrable). □

Proposition 3 *$u_x(\pi)$ is twice continuously differentiable in $\pi \in (0,1)$, we have $u'_x(\pi) = EL'_x(\pi)$ and $u''_x(\pi) = EL''_x(\pi)$ and these formulas hold true on the whole $[0,1]$, to which u'_x, u''_x can be continuously extended. We further have that $|u'_x(\pi)|, |u''_x(\pi)|$ are bounded, uniformly in x, π.*

Proof Using (9), (11), (12) and estimates analogous to those of Proposition 2 we get that $|L''_x(\pi)|, |L'_x(\pi)|$ are a.s. majorated by some random variable J, independent of x, π, which has finite expectation. For $\pi \in (0,1)$ and each (small enough) h we get, by the Lagrange mean value theorem,

$$\frac{L_x(\pi + h) - L_x(\pi)}{h} = L'_x(\xi(\pi, h))$$

for some $\xi = \xi(\pi, h)$ (depending also on $\omega \in \Omega$) between π and $\pi + h$. As the absolute value of the right-hand side is majorated by J, Lebesgue's theorem implies the differentiability of $u_x(\pi)$ at π as well as $u'_x(\pi) = EL'_x(\pi)$ and the claim about continuous extensions. The second derivative can be treated analogously. The last claim follows from

$$|u'_x(\pi)| \le EJ, \quad |u''_x(\pi)| \le EJ,$$

for all π, x. □

One can immediately see that $r(x) = u'_x(0)$ and this is a measurable (even continuous) function of x, bounded by Proposition 3. Define $\eta(x) := \max\{0, r(x)/2D\}$ where $D := EJ$ with J as in the previous proof. Note that $r(x) \le D$ and therefore $\eta(x) \le 1/2$, for all $x \in \mathbb{R}$.

Proposition 4 *If $r(x) > 0$ then $u_x(\eta(x)) \ge r^2(x)/(4D)$.*

Proof Indeed, as $u'_x(0) = r(x) > 0$ and $|u''_x(t)| \le EJ = D$ for all $t \in [0,1]$, we have that $u'_x(s) \ge r(x)/2$ whenever $s \in [0, \eta(x)]$, hence

$$u_x(\eta(x)) \ge \frac{\eta(x) r(x)}{2}$$

noting $u_x(0) = 0$. The statement follows. □

Proof (of Theorem 2) Let us define $\pi_t := \eta(X_{t-1})$. As pointed out in Remark 1, this is quite a natural choice: investing in the risky asset whenever it performs better than the riskless one. How much to invest is a more delicate question and that's why we needed the previous analytic consideration about $\eta(x)$.

Calculate

$$\begin{aligned}
\frac{1}{T}\ln V_T^{\pi} = \frac{1}{T}\ln V_0 + \frac{1}{T}\sum_{i=1}^{T}\ln[1-\eta(X_{i-1})+\eta(X_{i-1})\exp(X_i - X_{i-1})] = \\
\frac{1}{T}\ln V_0 + \frac{1}{T}\sum_{i=1}^{T}\{\ln[1-\eta(X_{i-1})+\eta(X_{i-1})\exp(X_i - X_{i-1})] - \\
-E[\ln[1-\eta(X_{i-1})+\eta(X_{i-1})\exp(X_i - X_{i-1})]|\mathcal{F}_{i-1}]\} + \\
+\frac{1}{T}\sum_{i=1}^{T}E[\ln[1-\eta(X_{i-1})+\eta(X_{i-1})\exp(X_i - X_{i-1})]|\mathcal{F}_{i-1}] = \\
\frac{1}{T}\ln V_0 + \frac{1}{T}\sum_{i=1}^{T}N_i + \frac{1}{T}\sum_{i=1}^{T}E[\ln[1-\eta(X_{i-1})+\eta(X_{i-1})\exp(X_i - X_{i-1})]|\mathcal{F}_{i-1}] \quad (15)
\end{aligned}$$

where we have set

$$\begin{aligned}
N_i := \ln[1-\eta(X_{i-1})+\eta(X_{i-1})\exp(X_i - X_{i-1})] - \\
-E[\ln[1-\eta(X_{i-1})+\eta(X_{i-1})\exp(X_i - X_{i-1})]|\mathcal{F}_{i-1}].
\end{aligned}$$

The conditional expectations exist by the estimations of Proposition 2. The N_i are martingale differences by definition. Estimations of Proposition 2 and square integrability of ε_1 imply $EN_i^2 \leq M$ where M is a constant independent of i. Theorem 2.15 on p.33 of [7] together with Kronecker's lemma (p.31 of [7]) entail that

$$\frac{1}{T}\sum_{i=1}^{T}N_i \to 0 \text{ a.s.}, \quad (16)$$

i.e. the law of large numbers applies to the first sum on the right-hand side. Let us further notice that, using the Markov property of X_t,

$$\begin{aligned}
E[\ln[1-\eta(X_{i-1})+\eta(X_{i-1})\exp(X_i - X_{i-1})]|\mathcal{F}_{i-1}] = \\
E[\ln[1-\eta(X_{i-1})+\eta(X_{i-1})\exp(X_i - X_{i-1})]|X_{i-1}] = u_{X_{i-1}}(\pi_i) \geq \\
\geq \frac{r^2(X_{i-1})}{4D}1_{\{r(X_{i-1})>0\}},
\end{aligned}$$

by Proposition 4, hence (10) and (16) imply that

$$\lim_{T\to\infty} P\left(\frac{1}{T}\ln V_T^{\pi} < \frac{c}{8D}\right) = 0,$$

from which the Theorem follows with the choice $\gamma := c/8D$. □

The precise analogue of Theorem 1 in the present context is the following.

Corollary 1 *Under the conditions of Theorem 2, there exists* $\gamma > 0$, $T_\varepsilon \in \mathbb{N}$ *such that for all* $T \geq T_\varepsilon$ *there are trading strategies* $\pi_t(\varepsilon, T), t \geq 1$ *satisfying* $V_T^{\pi(\varepsilon,T)} \geq V_0 - e^{-\gamma T/2}$ *and*

$$P(V_T^{\pi(\varepsilon,T)} \geq e^{\gamma T/2}) \geq 1 - \varepsilon. \tag{17}$$

Proof This is clear from Proposition 1. □

Criterion (10) of Theorem 2 depends on the law of ε_1 (since $r(x)$ does) which shows that while (2) works for all diffusion models, there are infinitely many versions of (10) and Theorem 2 corresponding to different choices for the law of ε_1.

Example 1 Take $\varepsilon_1 = \pm 1$ with probabilities $1/2 - 1/2$. Then

$$r(x) = \frac{1}{2}[e^{\mu(x)+\sigma(x)} + e^{\mu(x)-\sigma(x)}] - 1.$$

Example 2 When $\varepsilon_1 \sim N(0,1)$ we have

$$r(x) = e^{\mu(x)+\frac{\sigma^2(x)}{2}} - 1 \geq \mu(x) + \frac{\sigma^2(x)}{2}$$

whenever this latter is ≥ 0 (using $e^u \geq 1 + u$ for $u \geq 0$). It follows that if for some $c > 0$

$$\lim_{T\to\infty} P\left(\frac{1}{T}\sum_{i=1}^{T}\left(\mu(X_{i-1}) + \frac{\sigma^2(X_{i-1})}{2}\right)^2 1_{\{\mu(X_{i-1})+\frac{\sigma^2(X_{i-1})}{2}>0\}} < c\right) = 0, \tag{18}$$

one has AEA. We will sharpen this result in Theorem 3 below.

We now give a condition that ensures the validity of (10), compare to the discussion after Definition 1 above.

Proposition 5 *Assume that* σ, μ *are bounded and* (9) *holds. If* X_t *is a positive Harris chain with invariant measure* ν *and* $\nu(\{x : r(x) > 0\}) > 0$ *then* (10) *holds.*

Proof By the hypotheses, $\zeta := \int_{\mathbb{R}} F(x)\nu(dx)/2 > 0$ where $F(x) := r^2(x)1_{\{r(x)>0\}}$ is bounded and measurable. Apply the law of large numbers for Markov chains (Theorem 17.0.1 of [9]) to get

$$\frac{1}{T}\sum_{i=1}^{T} F(X_{i-1}) \to 2\zeta$$

a.s., hence

$$\lim_{T\to\infty} P\left(\frac{1}{T}\sum_{i=1}^{T} r^2(X_{i-1})1_{\{r(X_{i-1})>0\}} < \zeta\right) = 0,$$

showing (10).

Remark 3 One can check that, under the conditions of Theorem 4 below, X_t is positive Harris with ν equivalent to the Lebesgue measure on $\mathbb{R}$ (see Lemma 2 below), hence Proposition 5 applies whenever $r(x) > 0$ on a set of positive Lebesgue-measure.

One can, however, provide further examples: e.g. when $X_0, \mu, \sigma, \varepsilon_1$ are integer-valued, the effective state space of X_t is $\mathbb{Z}$ and ν is absolutely continuous w.r.t the counting measure. Proposition 5 is applicable if X_t is positive recurrent (this time X_t is a countable state space chain hence positive recurrence is the same as positive Harris recurrence, see p. 200 of [9]). We do not engage in a more detailed discussion here.

We investigate the Gaussian case further.

Theorem 3 *Assume that μ, σ are bounded, ε_1 is standard Gaussian, $\sigma > 0$ and for some $c > 0$,*

$$\lim_{T\to\infty} P\left(\frac{1}{T}\sum_{i=1}^{T}\left(\frac{\mu(X_{i-1})}{\sigma(X_{i-1})}+\frac{\sigma(X_{i-1})}{2}\right)^2 1_{\{\frac{\mu(X_{i-1})}{\sigma(X_{i-1})}+\frac{\sigma(X_{i-1})}{2}>0\}} < c\right) = 0. \tag{19}$$

Then there is AEA.

Note that, as σ is bounded, (19) is a weaker condition than (18) (assuming $\sigma > 0$).

Lemma 1 *If ε_1 is standard Gaussian then $|u''_x(\pi)| \le G\sigma^2(x)$ for some $G > 0$, for all $x \in \mathbb{R}$ and for all $0 \le \pi \le 1/2$.*

Proof For $0 \le \pi \le 1/2$ we have $|u''_x(\pi)| \le 4E[e^{\mu(x)+\sigma(x)\varepsilon_1} - 1]^2$, as directly verifiable using (12).

One can compute

$$E[e^{\mu(x)+\sigma(x)\varepsilon} - 1]^2 = e^{2\mu(x)+2\sigma^2(x)} - 2e^{\mu(x)+\sigma^2(x)/2} + 1 =$$
$$(e^{2\mu(x)+2\sigma^2(x)} - 1) - 2(e^{\mu(x)+\sigma^2(x)/2} - 1).$$

Fix $0 \le m \le 2K$ where K is a bound for both $|\mu(x)|$ and $|\sigma(x)|$. We consider a Taylor-expansion of $e^{m+s} - 1$ in $0 \le s < 1$:

$$e^{m+s} - 1 = m + s + R(s)$$

where the remainder term $R(s)$ satisfies

$$|R(s)| \le \frac{s^2}{2}\sup_{0\le t\le 1} e^{m+t}.$$

Hence

$$|R(s)| \le Vs^2 \le Vs,$$

for some constant $V := (1/2)e^{2K+1} < \infty$ and for $0 \le s < 1$.

It follows that for $0 \le \sigma(x) < 1$,

$$E[e^{\mu(x)+\sigma(x)\varepsilon} - 1]^2 \le |2\mu(x)+2\sigma^2(x)-2(\mu(x)+\sigma^2(x)/2)| + V\sigma^2(x) = \sigma^2(x) + V\sigma^2(x).$$

If $\sigma(x) \ge 1$ then

$$E[e^{\mu(x)+\sigma(x)\varepsilon} - 1]^2 \le E[e^{K+K|\varepsilon|} + 1]^2 =: H < \infty$$

by (9). Obviously, $H \le H\sigma^2(x)$ for $\sigma(x) \ge 1$.

It follows that, for all x,

$$E[e^{\mu(x)+\sigma(x)\varepsilon} - 1]^2 \le \max\{1+V, H\}\sigma^2(x),$$

showing the Lemma. □

Proof (of Theorem 3) Using Lemma 1 we may repeat the same proof as for Theorem 2, but defining $\eta(x) := \min\{\max\{r(x)/(2G\sigma^2(x)), 0\}, 1/2\}$. We get that

$$\lim_{T\to\infty} P\left(\frac{1}{T}\sum_{i=1}^{T}\frac{r^2(X_{i-1})}{\sigma^2(X_{i-1})}1_{\{r(X_{i-1})>0\}} < c\right) = 0$$

for some $c > 0$ implies AEA. As already pointed out in Example 2,

$$r(x) \geq \mu(x) + \frac{\sigma^2(x)}{2}$$

whenever $r(x) \geq 0$, so (19) indeed implies AEA and we may conclude. □

To round up the present section, we should compare Theorem 3 to the results of [5], in particular to Theorem 1 above. Since Brownian motion has Gaussian increments, when ε_1 is Gaussian, (5) can be regarded as a standard discretization of the SDE for $\ln S_t$ where S_t is positive and satisfies (1).

To make a reasonable comparison we should consider the case where $S_t = \exp\{X_t\}, t \in [0, \infty)$ for some X_t satisfying

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t.$$

Ito's formula gives us

$$dS_t = S_t\mu(\log S_t)dt + S_t\sigma(\log S_t)dW_t + \frac{1}{2}S_t\sigma^2(\log S_t)dt.$$

From this we get that the market price of risk is

$$\phi(S_t) = \frac{\mu(\log S_t)}{\sigma(\log S_t)} + \frac{\sigma(\log S_t)}{2}.$$

We can write ϕ as a function of X_t and get

$$\phi(X_t) = \frac{\mu(X_t)}{\sigma(X_t)} + \frac{\sigma(X_t)}{2},$$

hence (2) takes the form:

$$\lim_{T\to\infty} P\left(\frac{1}{T}\int_0^T \left(\frac{\mu(X_t)}{\sigma(X_t)} + \frac{\sigma(X_t)}{2}\right)^2 dt < c\right) = 0. \tag{20}$$

Now the analogy with (19) is straightforward, we only need to account for the indicators $1_{\{\frac{\mu(X_{i-1})}{\sigma(X_{i-1})}+\frac{\sigma(X_{i-1})}{2}>0\}}$, see Remark 2 above.

3 Control of failure probabilities

Throughout this section we assume that $\sigma(x), \mu(x)$ are bounded measurable functions and ε_1 satisfies

$$Ee^{a|\varepsilon_1|} < \infty$$

for all $a > 0$.

We recall a large deviation principle for martingales, a trivial corollary of Corollary 2.2 in [10].

Proposition 6 *Let $M_t, t \geq 1$ a martingale difference sequence (w.r.t. $\mathcal{F}_t, t \geq 0$) such that*

$$E[e^{|M_t|}|\mathcal{F}_{t-1}] \leq K \tag{21}$$

for some $K > 0$, for all $t \geq 1$. Then for all $\epsilon > 0$ there are $c = c_\epsilon > 0$ and $C = C_\epsilon > 0$ such that for all $T \geq 1$,

$$P\left(\left|\frac{\sum_{i=1}^T M_i}{T}\right| \geq \epsilon\right) \leq Ce^{-cT}.$$

Theorem 4 *Under the conditions of the present section,*

$$\limsup_{T\to\infty} \frac{1}{T} \ln P\left(\frac{1}{T}\sum_{i=1}^T r^2(X_{i-1})1_{\{r(X_{i-1})>0\}} \leq c_1\right) \leq -c_2. \tag{22}$$

for some $c_1, c_2 > 0$ implies AEA with geometrically decaying failure probabilities.

Proof We again set $\pi_i := \eta(X_{i-1})$, as in the proof of Theorem 2. Define $H_t := \ln(1 - \eta(X_{t-1}) + \eta(X_{t-1})\exp\{X_t - X_{t-1}\})$. Note that, for some $C > 0$, independent of t,

$$E[\exp\{H_t - E[H_t|\mathcal{F}_{t-1}]\}|\mathcal{F}_{t-1}] = e^{-E[H_t|X_{t-1}]}E[e^{H_t}|X_{t-1}] \leq$$
$$e^{E[|H_t|\,|X_{t-1}]}E[e^{|H_t|}|X_{t-1}] \leq (E[e^{|H_t|}|X_{t-1}])^2 \leq (Ee^{C(1+|\varepsilon_1|)})^2 < \infty$$

by the Markov property, Jensen's inequality and the estimates of Proposition 2. Recall $N_t = H_t - E[H_t|\mathcal{F}_{t-1}]$ from (15) and conclude that the martingale differences N_t satisfy the conditions of Proposition 6, hence

$$P\left(\left|\frac{\sum_{i=1}^T N_i}{T}\right| \geq \frac{c_1}{8D}\right) \leq Ce^{-cT}. \tag{23}$$

for suitable c, C and with D defined before Proposition 4. Now we have

$$\frac{1}{T}\ln V_T^\pi = \frac{1}{T}\ln V_0 + \frac{1}{T}\sum_{i=1}^T N_i +$$
$$+\frac{1}{T}\sum_{i=1}^T E[\ln[1 - \eta(X_{i-1}) + \eta(X_{i-1})\exp(X_i - X_{i-1})]|\mathcal{F}_{i-1}] \geq$$
$$\frac{1}{T}\ln V_0 + \frac{1}{T}\sum_{i=1}^T N_i + \frac{1}{T}\sum_{i=1}^T \frac{r^2(X_{i-1})}{4D}1_{\{r(X_{i-1})>0\}},$$

hence for $\gamma := \min\{c, c_2\}$ and $\beta := c_1/(8D)$ we get

$$\limsup_{T\to\infty} \frac{1}{T} \ln P\left(\frac{1}{T}\ln V_T^\pi \le \beta\right) \le -\gamma,$$

proving the Theorem. □

Corollary 2 *Under the conditions of Theorem 4, there exists $\gamma, C > 0$, $T_\varepsilon \in \mathbb{N}$ such that for all $T \ge T_\varepsilon$ there are trading strategies $\pi_t(\varepsilon, T), t \ge 1$ satisfying $V_T^{\pi(\varepsilon,T)} \ge V_0 - e^{-\gamma T/2}$ and*

$$P(V_T^{\pi(\varepsilon,T)} \ge e^{\gamma T/2}) \ge 1 - Ce^{-\gamma T}. \tag{24}$$

Proof The argument of Proposition 1 applies again. □

Theorem 5 *In addition to the hypotheses of the present section, let us assume that the law of ε_1 is absolutely continuous with respect to the Lebesgue measure with a density $h(u), u \in \mathbb{R}$ that is bounded away from 0 on compacts. Assume further that $\sigma(x)$ is bounded away from 0 on compacts and $\{x \in \mathbb{R} : r(x) > 0\}$ has positive Lebesgue-measure. If there is $V : \mathbb{R} \to [1, \infty)$ such that for all $x \in \mathbb{R}$,*

$$E[V(X_1)|X_0 = x] \le (1 - \delta)V(x)1_{\{x \notin \Gamma\}} + C1_{\{x\in\Gamma\}} \tag{25}$$

for a bounded interval $\Gamma := [e, f]$, $e < f$ and for some $0 < \delta < 1$, $C > 0$ then (22) holds true and hence there is AEA with geometrically decaying failure probabilities.

Lemma 2 *Under the conditions of Theorem 5, the Markov chain X_t is Lebesgue-irreducible and aperiodic; intervals $[e, f]$ with $e < f$ are small sets and X_t is geometrically ergodic. X_t is a positive Harris chain with invariant measure $\nu \sim Leb$, where Leb denotes the Lebesgue measure on $\mathbb{R}$.*

Proof All these notion are defined in [9] on pages 82, 102, 114, 199, 231, 363. Let $A \subset \mathbb{R}$ have positive Lebesgue-measure. Then

$$P(X_1 \in A|X_0 = x) = P\left(\varepsilon_1 \in \frac{A - \mu(x) - x}{\sigma(x)}\right) = \int_{\frac{A-\mu(x)-x}{\sigma(x)}} h(u)du.$$

As $\sigma(x) \neq 0$ and the Lebesgue-measure is translation invariant,

$$Leb\left(\frac{A - \mu(x) - x}{\sigma(x)}\right) > 0,$$

showing $P(X_1 \in A|X_0 = x) > 0$ and hence Lebesgue-irreducibility of the chain. Aperiodicity is also straightforward as $P(X_1 \in A|X_0 = x)$ is a one-step transition probability.

If $x \in [e, f]$ then

$$P(X_1 \in [e, f]|X_0 = x) = \int_{\frac{[e,f]-\mu(x)-x}{\sigma(x)}} h(u)du$$

and note that, as $\mu(x)$ is bounded and $\sigma(x)$ is bounded from below on $[e, f]$, the set $\cup_{x\in[e,f]} \frac{[e,f]-\mu(x)-x}{\sigma(x)}$ is bounded and hence $h(u)$ is bounded from below on it. This shows that $[e, f]$ is a small set. The drift condition (25) implies geometric ergodicity, see Theorem 15.0.1 of [9], since Γ is a small set.

Harris recurrence follows from (25) and Theorem 9.1.8 of [9]. An invariant measure ν exists by Theorem 15.0.1, showing positivity. As $P(X_1 \in \cdot | X_0 = x)$ is Lebesgue absolutely continuous for each x, we get $\nu \ll Leb$. Theorem 10.4.9 of [9] implies $Leb \ll \nu$. □

Proof (of Theorem 5.) Define $F(u) := r^2(u)1_{\{r(u)>0\}}$. This is bounded and measurable and $\nu \sim Leb$ implies that $z = \int_{\mathbb{R}} F(u)\nu(du) > 0$. The chain X_t is Lebesgue-irreducible, aperiodic and geometrically ergodic by Lemma 2 above. One may always assume that $\int_{\mathbb{R}} V^2(x)\nu(dx) < \infty$, see Theorem 14.0.1 and Lemma 15.2.9 of [9]. Hence

$$\rho^2 := \lim_{t\to\infty} \frac{1}{t}[F(X_0) + \ldots + F(X_{t-1})]$$

is well defined, see p. 317 of [8]. If $\rho^2 = 0$ then (i) of Proposition 2.4 in [8] shows that F is Lebesgue a.s. constant. In this case Theorem 5 follows trivially. Hence we may and will assume $\rho^2 > 0$.

Theorem 4.1 and P4 on page 343 from [8] show that there is $\theta > 0$ and an analytic function $\Lambda(\alpha)$, $\alpha \in (z-\theta, z+\theta)$. such that

$$\lim_{t\to\infty} \frac{1}{t} \ln Ee^{F(X_0)+\ldots+F(X_{t-1})} = \Lambda(\alpha)$$

and $\Lambda''(\alpha) = \rho^2 > 0$. We may assume that θ is so small that $\Lambda''(\alpha) > 0$ for $\alpha \in (z-\theta, z+\theta)$, hence $I(\beta) := (\Lambda')^{-1}(\beta)$ is well-defined for $\beta \in (\Lambda'(z-\theta), \Lambda'(z+\theta)) =: (\underline{b}, \overline{b})$. Then the Legendre-transform

$$\Lambda^*(\beta) := \sup_{\alpha\in(z-\theta,z+\theta)} [\beta\alpha - \Lambda(\alpha)]$$

can be written as $\Lambda^*(\beta) = \beta I(\beta) - \Lambda(I(\beta))$ for $\beta \in (\underline{b}, \overline{b})$ and one may check that $(\Lambda^*)''(\beta) = 1/\Lambda''(I(\beta)) > 0$ for $\beta \in (\underline{b}, \overline{b})$ showing the strict convexity of Λ^*. As easily seen, $\Lambda^*(\beta) \geq 0$ for all $\beta \in (\underline{b}, \overline{b})$ and $\Lambda^*(z) = 0$ hence for all $\kappa \in (z-\theta, z)$, $\Lambda^*(\kappa) > 0$.

Theorem 4.1 of [8] and the Gärtner-Ellis theorem (see Theorem 2.3.6 in [1]) guarantee that the following large deviation principle holds:

$$P\left(\frac{\sum_{i=1}^T F(X_{i-1})}{T} < \kappa\right) \leq Ce^{-T\Lambda^*(\kappa)},$$

for some $C > 0$. This shows that (22) holds true and then Theorem 4 allows us to conclude.

The next Proposition shows that X_t satisfies (25) provided that the drift $\mu(x)$ is "mean-reverting enough".

Proposition 7 *There exists $M > 0$, depending on σ, ε_1 such that X_t satisfies* (25) *provided that there are $N_+, N_- > 0$ such that*

$$\mu(x) \leq -M \text{ for } x \geq N_+$$

and

$$\mu(x) \geq M \text{ for } x \leq -N_-.$$

Proof Let K_σ, K_μ denote a bounds for $|\sigma|, |\mu|$, respectively. Let us take the Lyapunov function $V(x) := e^{|x|}$ and note

$$E[V(X_1)|X_0 = x] \le e^{|x+\mu(x)|}L_1 = e^{x+\mu(x)}L_1$$

for $x \ge K_\mu$ with $L_1 := Ee^{K_\sigma|\varepsilon_1|}$. Similarly,

$$E[V(X_1)|X_0 = x] \ge e^{|x+\mu(x)|}L_2 = e^{x+\mu(x)}L_2$$

for $x \le -K_\mu$ with $L_2 = Ee^{-K_\sigma|\varepsilon_1|}$. Let $M := 1 + \max\{\ln L_2, -\ln L_1\}$, take N_-, N_+ as in the hypothesis. Define

$$\Gamma := [\min\{-N_-, -K_\mu\}, \max\{N_+, K_\mu\}].$$

We can see that for $x \notin \Gamma$,

$$E[V(X_1)|X_0 = x] \le (1-\delta)V(x)$$

for some $\delta > 0$. It is clear that for all $x \in \Gamma$,

$$E[V(X_1)|X_0 = x] \le C$$

for a suitable $C > 0$, hence (25) holds. □

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