Initial blow-up solution of a semilinear heat equation

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Abstract

We study the existence and uniqueness of a maximal solution of equation $u_t - \Delta u + f(u) = 0$ in $\Omega \times (0, \infty)$, where Ω is a domain with a non-empty compact boundary, which satisfies $u = g$ on $\partial\Omega \times (0, \infty)$, assuming that g and f are given continuous functions and f is also convex, nondecreasing, $f(0) = 0$ and verifies Keller-Osserman condition. We show that if the boundary of Ω satisfies the parabolic Wiener criterion then the maximal solution is the unique large solution, i. e., it blows up at $t = 0$.

Key words: semilinear heat equation, maximal monotone operators, Wiener criterion, maximal and large solutions.

1. Introduction

We consider the following semilinear parabolic equation :

$$u_t - \Delta u + f(u) = 0 \text{ in } Q_\Omega^\infty := \Omega \times (0, \infty), \tag{1}$$

with the boundary condition

$$u = g \text{ on } \partial_l Q_\Omega^\infty = \partial\Omega \times (0, \infty), \tag{2}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a domain (bounded or possibly unbounded) with a non-empty compact boundary. Concerning the nonlinear functions f and g we assume that

$$f : [0, \infty) \to [0, \infty) \text{ is a continuous and non-decreasing function,}$$

$$f(0) = 0, \; f > 0 \text{ on } (0, \infty), \tag{f_1}$$

which satisfies the Keller-Osserman condition (see [6], [9]):

$$\int_1^\infty [2F(s)]^{-1/2} ds < \infty \text{ where } F(s) = \int_0^s f(\tau) d\tau, \tag{f_2}$$

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and $g : \partial_l Q_\Omega^\infty \to [0, \infty)$ is a continuous given function.
The main purpose of this paper is to study the existence and uniqueness of a *large initial solution* of equation (1), subject to the boundary condition (2), that is a positive function u which satisfies (1), (2) and

$$\lim_{t \to 0} u(\cdot, t) = \infty, \quad \text{uniformly on any compact subset of } \Omega. \tag{3}$$

The existence of large solutions for the corresponding elliptic equation

$$-\Delta u + f(u) = 0 \text{ in } \Omega, \tag{4}$$

has been originally studied by Keller [6] and Osserman [9] in 1957. In these works they provided a condition on f (more precisely, condition (f_2)) which is necessary and sufficient in order that the set of solutions of (4) should be locally bounded from above. The result of Keller and Osserman states:

Theorem 1. *Let f be a function satisfying assumptions (f_1) and (f_2). There exists a continuous and nonincreasing function h defined on $\mathbb{R}_+$ with the limits*

$$\lim_{\rho \to 0} h(\rho) = \infty, \quad \lim_{\rho \to \infty} h(\rho) = 0,$$

such that, for any domain $\Omega \subsetneq \mathbb{R}^N$ and any function $u \in C(\Omega)$ satisfying $\Delta u \geq f(u)$ in $\mathcal{D}'(\Omega)$, $u(x) \leq h(dist(x, \partial\Omega))$ $\forall x \in \Omega$.

As a consequence of the above Theorem, it is known that if f satisfies conditions (f_1) and (f_2) then a large solution u (i. e. a function $u \in C^1(\Omega)$ which verifies equation (4) in Ω and $u(x) \to \infty$ as $\text{dist}(x, \partial\Omega) \to 0$, $x \in K$, for every bounded set $K \subset \Omega$) exists in every bounded domain which possesses a Lipschitz boundary. Labutin [5] showed that the Lipschitz condition on $\partial\Omega$ can be replaced by a Wiener type condition which is necessary and sufficient for the existence of a large solution. Uniqueness in smooth domains was established under some additional assumptions on f (see [2]). Other results concerning existence and uniqueness for large solutions of (4) in non-smooth domains Ω have been obtained by Marcus and Veron in [7].

The aim of this article is to extend these questions to the parabolic equation (1). We note that the special case $f(u) = \mid u \mid^{q-1} u$, $q > 1$, has been studied by Waad Al Sayed and Veron in [1]. It is worth pointing out that the existence and the uniqueness of the solution of problem (1) which satisfies $u = \infty$ on the parabolic boundary $\overline{\Omega} \times \{0\} \bigcup \partial\Omega \times (0, \infty)$ when Ω is a domain in $\mathbb{R}^N$ with a compact boundary and f is a continuous increasing function satisfying superlinear growth condition have been studied by Marcus and Veron in a recent article [8].

In Section 2 we consider the following problem, denoted by $P_{\Omega,0}$:

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q_\Omega^\infty, \\ u = 0 & \text{on } \partial_l Q_\Omega^\infty. \end{cases} \tag{$P_{\Omega,0}$}$$

The first result we obtain in this paper is an existence theorem for maximal solutions of problems $P_{\Omega,0}$ under appropriate assumptions on function f, when Ω is an arbitrary subdomain of $\mathbb{R}^N$ with compact boundary (see Theorem 8 below). Next, we prove that a such solution is also a minimal solution of our problem which satisfies (3) (Proposition 9). In the next part of this Section we suppose that $\partial\Omega$ verifies the parabolic Wiener criterion. In this case the solution constructed in Theorem 8 is more regular, as we shall prove in Theorem 12 (see also Proposition 13). The last purpose of the paper is to obtain a result concerning the existence and uniqueness for the solutions of problem consisting of equation (1), lateral boundary condition (2) and (3). The assumptions we should require to achieve our goal are given in Theorem 14, the main result of Section 3. We point out that the existence and uniqueness of such a solution u is associated to the existence of a large solution to the stationary equation (4) and solution of the ODE (see Remark 6)

$$\varphi' + f(\varphi) = 0 \ \text{ in } \ (0,\infty), \ \lim_{t\searrow 0} \varphi(t) = \infty.$$

2. Maximal solutions of problem $P_{\Omega,0}$

We will be concerned with problem $P_{\Omega,0}$ formulated above under assumptions $(f_1),(f_2)$, where Ω is a domain of $\mathbb{R}^N$, $N \geq 2$, with a non-empty compact boundary.

In order to investigate this problem, we choose as our framework the Hilbert space $H := L^2(\Omega)$, endowed with the usual scalar product, denoted $\langle \cdot,\cdot \rangle$, and the corresponding induced norm, denoted $\|\cdot\|$. It is not difficult to see that $P_{\Omega,0}$ can be represented as an equation in the Hilbert space H, associated to a maximal monotone operator:

$$\frac{du}{dt}(t) + A_\Omega(u(t)) = 0, \ t \in (0,\infty), \tag{5}$$

where $u(t) := u(\cdot,t)$, $\overline{f} : D(\overline{f}) \subset H \to H$,

$$D(\overline{f}) = \{u \in H; \ x \to f(u(x)) \text{ belongs to } H\},$$
$$\overline{f}(u)(x) = f(u(x)) \text{ for a. a. } x \in \Omega, \ \forall\, u \in D(\overline{f}),$$

($\overline{f}$ is the canonical extension of f to H), $A_\Omega : D(A_\Omega) \subset H \to H$,

$$D(A_\Omega) = \{u \in H_0^1(\Omega); \ \Delta u \in H\} \bigcap D(\overline{f}), \ A_\Omega u = -\Delta u + \overline{f}(u) \ \forall\, u \in D(A_\Omega).$$

It is well known that under assumption (f_1), operator A_Ω is maximal monotone. In fact, A_Ω is even cyclically monotone. More precisely, since f is the derivative of the convex function $j(x) = \int_0^x f(y)dy$, we have $A_\Omega = \partial\psi_\Omega$, where $\psi_\Omega : H \to (-\infty,\infty]$,

$$\psi_\Omega(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega j(u)dx, \ \text{if } u \in H_0^1(\Omega), \ j(u) \in L^1(\Omega), \\ +\infty, \ \text{otherwise}, \end{cases}$$

where $|\cdot|$ denotes the euclidian norm of $\mathbb{R}^N$ (see, e.g., [10], p. 197). Therefore, according to well known Brezis' result (see [3])), we have:

Theorem 2. *Assume that* (f_1) *is satisfied and* $u_0 \in H$ *is a given function. Then, there exists a unique strong solution* u *of equation* (5) *which satisfies the initial condition* $u(0) = u_0$*, such that* $u(t) \in D(A_\Omega)\ \forall\ t > 0$,

$$u \in C([0, \infty); L^2(\Omega)) \bigcap L^2_{loc}(0, \infty; H^1_0(\Omega)) \bigcap H^1_{loc}(0, \infty; L^2(\Omega)).$$

In addition,

$$\| (du/dt)(t) \| \leq 1/t \| u_0 \| \quad \forall\ t \in (0, \infty). \tag{6}$$

In the following we consider solutions of equation(5) with the above regularity.

Definition 3. *Let* Ω *be a bounded subdomain of* $\mathbb{R}^N$*. We denote by* $\mathcal{S}_0(Q^\infty_\Omega)$ *the set of positive functions* $u \in C(0, \infty; L^2(\Omega)) \bigcap L^2_{loc}(0, \infty; H^1_0(\Omega))$ *such that* $u_t,\ \Delta u \in L^2_{loc}(0, \infty; L^2(\Omega)),\ u(\cdot, t) \in D(A_\Omega)\ \forall\ t > 0$*, which verify* (5) *for all* $t > 0$.

More generally, we denote by $\mathcal{S}(Q^\infty_\Omega)$ *the set of all positive functions* $u \in L^2_{loc}(0, \infty; H^1(\Omega)) \bigcap H^1_{loc}(0, \infty; L^2(\Omega)),\ f(u) \in L^2_{loc}(0, \infty; L^2(\Omega))$*, such that*

$$\int_\Omega [(\partial u/\partial t)\zeta + \nabla u \cdot \nabla \zeta + f(u)\zeta](\cdot, t)dx = 0\ \forall\ t > 0,\ \zeta \in H^1_0(\Omega). \tag{7}$$

We define in a similar way a subsolution (resp. supersolution) of (1) *by imposing the same regularity conditions on* u *and* $f(u)$ *and*

$$\int_\Omega [(\partial u/\partial t)\zeta + \nabla u \cdot \nabla \zeta + f(u)\zeta](\cdot, t)dx \leq 0\ \forall\ t > 0,\ \zeta \in H^1_0(\Omega), \tag{8}$$

respectively,

$$\int_\Omega [(\partial u/\partial t)\zeta + \nabla u \cdot \nabla \zeta + f(u)\zeta](\cdot, t)dx \geq 0\ \forall\ t > 0,\ \zeta \in H^1_0(\Omega). \tag{9}$$

Now we consider the case when Ω is unbounded and we assume that $\Omega^c \subset B_{R_0}$, denote by $\Omega_R = \Omega \bigcap B_R,\ R \geq R_0$, and by $\widetilde{H}^1_0(\Omega_R)$ the closer in $H^1(\Omega_R)$ of the space of $C^\infty(\Omega_R)-$ functions which vanish in a neighborhood of $\partial\Omega$.

Definition 4. *If* Ω *is not bounded and* $\Omega^c \subset B_{R_0}$*, we denote by* $\mathcal{S}_0(Q^\infty_\Omega)$ *the set of positive functions* u *such that, for any* $R \geq R_0$, $u \in C(0, \infty; L^2(\Omega_R)) \bigcap L^2_{loc}(0, \infty; \widetilde{H}^1_0(\Omega_R)),\ u_t, \Delta u \in L^2_{loc}(0, \infty; L^2(\Omega_R)),\ u(\cdot, t) \in \widetilde{H}^1_0(\Omega_R)\ \forall\ t > 0$, *which verify* (5) *for all* $t > 0$*. Also, we denote by* $\mathcal{S}(Q^\infty_\Omega)$ *the set of all positive functions*

$$u \in L^2_{loc}(0, \infty; H^1(\Omega_R)) \bigcap H^1_{loc}(0, \infty; L^2(\Omega_R)),\ f(u) \in L^2_{loc}(0, \infty; L^2(\Omega_R)),$$

such that

$$\int_{\Omega_R} [(\partial u/\partial t)\zeta + \nabla u \cdot \nabla \zeta + f(u)\zeta](\cdot, t)dx = 0\ \forall\ t > 0,\ \zeta \in H^1_0(\Omega_R), \tag{10}$$

for all $R \geq R_0$.

Remark 5. Suppose that Ω is a bounded subdomain of $\mathbb{R}^N$, f verifies assumption (f_1), and

$$f \text{ is a convex function.} \tag{f_3}$$

Obviously, if $u_1,\ u_2 \in \mathcal{S}(Q_\Omega^\infty)$, then $u_1 + u_2$ is a supersolution of (1).

On the other hand, if $u \in \mathcal{S}(Q_\Omega^\infty)$ and v is a supersolution of (1), such that supp $[u-v]_+(\cdot,t)$ is a compact subset of $\Omega\ \forall\, t>0$ and $\lim_{t\searrow 0} \parallel [u-v]_+(\cdot,t) \parallel = 0$, then $u(x,t) \le v(x,t)\ \forall\, t>0$, for a.a. $x \in \Omega$.

Indeed, starting with the obvious inequality

$$\int_\Omega \Big(\frac{\partial(u-v)}{\partial t}\zeta + \nabla(u-v)\cdot\nabla\zeta \Big)(\cdot,t)dx \le \int_\Omega (f(v)-f(u))(\cdot,t)\zeta dx,$$

$\forall\ t>0,\ \zeta \in H_0^1(\Omega)$, and taking in the above inequality $\zeta = [u-v]_+(\cdot,t) \in H_0^1(\Omega)\ \forall\, t>0$, we can see that

$$\begin{aligned} \int_\Omega \Big(\frac{\partial h}{\partial t} h + \mid \nabla h \mid^2 \Big)(\cdot,t)dx &\le \int_\Omega \big((f(v)-f(u))h \big)(\cdot,t)dx \\ &\le -\int_\Omega \big(f(h)h \big)(\cdot,t)dx \le 0\ \forall\, t>0, \end{aligned} \tag{11}$$

where $h := [u-v]_+$ (we have used that f is a convex function, $f(0)=0$, therefore $f(a+b) \ge f(a)+f(b)\ \forall\ a,b \ge 0$).

Clearly, inequality (11) implies

$$\frac{d}{dt} \parallel h(\cdot,t) \parallel^2 \le 0\ \forall\, t>0 \Rightarrow\ t \to \parallel h(\cdot,t) \parallel \quad \text{is non-increasing on } (0,\infty).$$

Therefore, $\parallel h(\cdot,t) \parallel \le \lim_{s\searrow 0} \parallel h(\cdot,s) \parallel = 0$, which implies $h(x,t) = 0 \Leftrightarrow u(x,t) \le v(x,t)\ \forall\, t>0,$ for a.a. $x \in \Omega$.

Remark 6. If assumptions $(f_1)-(f_3)$ are satisfied, then there exists a unique solution $\varphi : (0,\infty) \to (0,\infty)$, of the following problem:

$$\begin{cases} \varphi' + f(\varphi) = 0 \ \text{ in } \ (0,\infty), \\ \lim\limits_{t\searrow 0} \varphi(t) = \infty. \end{cases} \tag{12}$$

Indeed, it is known that under assumptions $(f_1)-(f_3)$, $\widetilde{F}(t) := -\int_t^\infty \frac{1}{f(s)}ds < \infty\ \forall\ t>0$. On the other hand, since f is a convex function, $s \to f(s)/s$ is non-decreasing on $(0,\infty)$, therefore $f(s)/s \le f(1)\ \forall\ 0<s\le 1$. Thus for every $s \in (0,1)$ we have

$$\int_s^1 \frac{dt}{f(t)} \ge \frac{1}{f(1)} \int_s^1 \frac{dt}{t} = -\frac{1}{f(1)} \ln s.$$

It follows that $\lim_{s\searrow 0} \int_s^1 \frac{dt}{f(t)} = \infty$, which implies $\lim\limits_{t\searrow 0} \widetilde{F}(t) = -\infty$. Therefore, $\widetilde{F} : (0,\infty) \to (-\infty,0)$ is a C^1 increasing bijective function and the general solution of equation $(11)_1$ is $\widetilde{F}(\varphi(t)) = -t+K,\ K \le 0$. Taking into account $(12)_2$, we obtain $K=0$, therefore the unique solution of problem (12) is $\varphi(t) = \widetilde{F}^{-1}(-t)\ \forall\ t>0$.

Lemma 7. *Suppose that assumptions* $(f_1) - (f_3)$ *are satisfied. If* $u \in \mathcal{S}(Q_\Omega^\infty)$ *the following estimate holds:*

$$u(x,t) \leq \varphi(t) + U_\Omega(x) \quad \forall\ (x,t) \in Q_\Omega^\infty, \tag{13}$$

where $U_\Omega \in C^1(\Omega)$ *is a maximal solution of the equation*

$$-\Delta u + f(u) = 0 \quad in \quad \Omega \tag{14}$$

(for the existence of a such solution, see [7], p. 642).

If, in addition, Ω *is bounded and* $u \in \mathcal{S}_0(Q_\Omega^\infty)$*, then:*

$$u(x,t) \leq \varphi(t) \quad \forall\ (x,t) \in Q_\Omega^\infty. \tag{15}$$

Proof. Let $\{\Omega^n\}_{n\in\mathbb{N}}$ be a sequence of bounded subsets of Ω with smooth boundary such that $\overline{\Omega}^n \subset \Omega^{n+1}$, $\Omega = \bigcup_{n\in\mathbb{N}} \Omega^n$. For every $n \in \mathbb{N}$, denote by U_n the maximal solution of (14) in Ω^n (in fact, as Ω^n is bounded and possesses smooth boundary, U_n is a large solution). Let $u \in \mathcal{S}(Q_\Omega^\infty)$.

Clearly, $\varphi, U_n \in \mathcal{S}(Q_{\Omega^n}^\infty)$, therefore $\varphi + U_n$ is a supersolution of (1) in $Q_{\Omega^n}^\infty$. For $\tau > 0$ and $n \in \mathbb{N}$, we denote by $v_{n,\tau}$ the function defined as follows

$$v_{n,\tau}(x,t) := u(x,t+\tau) - (U_n(x) + \varphi(t))\ \forall\ (x,t) \in Q_{\Omega^n}^\infty.$$

Obviously, supp $[v_{n,\tau}]_+(\cdot,t)$ is a compact subset of $\Omega^n\ \forall\ t > 0$. In addition, by Lebesgue's theorem, $\lim_{t\searrow 0} \parallel [v_{n,\tau}]_+(\cdot,t) \parallel_{L^2(\Omega^n)} = 0$. Thus, making use of Remark 5, we find $u(x,t+\tau) \leq U_n(x) + \varphi(t)\ \forall\ t > 0,$ for a.a. $x \in \Omega^n$. Letting $\tau \searrow 0$ and using the continuity we find

$$u(x,t) \leq U_n(x) + \varphi(t)\ \forall\ t > 0,\ x \in \Omega^n. \tag{16}$$

Finally, we can pass to the limit as $n \to \infty$ in inequality (16) to obtain (13).

The proof of the last part of the Lemma goes similarly, with slight modifications. More exactly, if Ω is bounded and $u \in \mathcal{S}_0(Q_\Omega^\infty)$, we denote $v_\tau(x,t) := u(x,t+\tau) - \varphi(t)\ \forall\ (x,t) \in Q_\Omega^\infty$. Since supp $[v_\tau]_+(\cdot,t)$ is a compact subset of $\Omega\ \forall\ t > 0$ and, on account of Lebesgue's theorem, $\lim_{t\searrow 0} \parallel [v_\tau]_+(\cdot,t) \parallel = 0$, by Remark 5 we find that $u(x,t+\tau) \leq \varphi(t) \quad \forall\ t > 0,$ for a.a. $x \in \Omega$. Inequality (15) follows by letting $\tau \searrow 0$.

Next, we continue with a result about the existence of a maximal solution of problem $P_{\Omega,0}$:

Theorem 8. *Assume that* $(f_1) - (f_3)$ *are satisfied. Then problem* $P_{\Omega,0}$ *has a maximal solution* $\widehat{u}_\Omega \in \mathcal{S}_0(Q_\Omega^\infty)$*, more precisely:*

$$\widehat{u}_\Omega \geq u \ \ in \ Q_\Omega^\infty,\ \forall\ u \in \mathcal{S}_0(Q_\Omega^\infty). \tag{17}$$

Proof. Step 1. Construction of $\widehat{u}_\Omega$ in the case that Ω is bounded.

Let $k \in \mathbb{N}^*$. By Theorem 2, the following Cauchy problem in H:

$$\begin{cases} \frac{du_k}{dt}(t) + A_\Omega(u_k(t)) = 0, \ t \in (0,\infty), \\ u_k(0) = k \text{ in } \Omega, \end{cases} \tag{18}$$

has a unique strong solution, denoted by $u_k \in \mathcal{S}_0(Q_\Omega^\infty)$. Since f is non-decreasing, it follows by the maximum principle that the corresponding sequence of these solutions $\{u_k\}_k$ increases with k and in view of Lemma 7 (see (15)) is locally uniformly bounded in Q_Ω^∞ by φ. Thus, for every $(x,t) \in Q_\Omega^\infty$ we can define $\widehat{u}_\Omega(x,t) := \lim_{k\to\infty} u_k(x,t)$.

Next, we check that $\widehat{u}_\Omega \in \mathcal{S}_0(Q_\Omega^\infty)$ is a maximal solution of problem $P_{\Omega,0}$. Let K be a compact subset of $(0,\infty)$. Now, making use of inequalities (6), (15), assumption (f_1) and equation $(18)_1$ we find

$$\begin{aligned} &\| (du_k/dt)(\cdot,t) \| = \| A_\Omega(u_k)(\cdot,t) \| \le (1/t_0)\sqrt{\text{meas}(\Omega)}\varphi(t_0), \\ &\qquad \| u_k(\cdot,t) \| \le \sqrt{\text{meas}}(\Omega)\varphi(t_0), \ f(u_k(\cdot,t)) \le f(\varphi(t_0)) \ \forall\ k \in \mathbb{N}, \ t \in K, \end{aligned}$$

where $t_0 := \inf\{t;\ t \in K\}$. Therefore, according to Lebesgue's theorem and demi-closedness of maximal monotone operators we obtain that

$$\begin{aligned} &u_k \to \widehat{u}_\Omega \ \text{ in } H^1_{loc}(0,\infty;L^2(\Omega)), f(u_k) \to f(\widehat{u}_\Omega) \ \text{ in } L^2_{loc}(0,\infty;L^2(\Omega)), \\ &\widehat{u}_\Omega \in H^1_{loc}(0,\infty;L^2(\Omega)), \ f(\widehat{u}_\Omega) \in L^2_{loc}(0,\infty;L^2(\Omega)), \widehat{u}_\Omega(\cdot,t) \in D(A_\Omega), \\ &\quad A_\Omega(u_k(\cdot,t)) \rightharpoonup A(\widehat{u}_\Omega(\cdot,t)) \text{ in } \ L^2(\Omega) \ \forall\ t > 0. \end{aligned}$$

We let $k \to \infty$ in $(18)_1$ and derive that $\widehat{u}_\Omega$ verify (5) for all $t > 0$. Now, let us verify that $\widehat{u}_\Omega \in C(0,\infty;L^2(\Omega))$. By a standard computation, we find

$$\| (u_k - u_l)(\cdot,t) \|^2 + 2\int_s^t \| \nabla(u_k - u_l)(\cdot,\theta) \|^2_{(L^2(\Omega))^N} \, d\theta \le \| (u_k - u_l)(\cdot,s) \|^2,$$

$\forall\ 0 < s < t$. Since $\| (u_k - u_l)(\cdot,s) \| \to 0$ as $k,l \to \infty$, we obtain $u_k \to \widehat{u}_\Omega$ in $C([s,t];L^2(\Omega))$ $\forall\ 0 < s < t$ which implies $\widehat{u}_\Omega \in C(0,\infty;L^2(\Omega))$.

It remains to show that $\widehat{u}_\Omega$ is a maximal solution of problem $P_{\Omega,0}$. To this end, let $\tau > 0$, $u \in \mathcal{S}_0(Q_\Omega^\infty)$, $k_\tau \in \mathbb{N}$, such that $u_{k_\tau}(x,0) = k_\tau > \varphi(\tau)$ and define the function

$$v_\tau(x,t) := u(x,t+\tau) - u_{k_\tau}(x,t) \ \forall (x,t) \in Q_\Omega^\infty.$$

It is easy to see that $v_\tau(\cdot,t) \in H^1_0(\Omega)\ \forall\, t > 0$. In addition, $\lim_{t\searrow 0} \| [v_\tau]_+(\cdot,t) \| = 0$, and thus, using a similar reasoning as in Remark 5, $v_\tau(\cdot,t) = 0\ \forall\ t > 0$, a.e. in Ω. Finally, making use of the continuity and letting $\tau \to 0$ (which implies $k_\tau \to \infty$) it follows that $u \le \widehat{u}_\Omega$ in Q_Ω^∞.

Step 2. Construction of $\widehat{u}_\Omega$ when Ω is unbounded.

Let $R_0 > 0$ be large enough such that $\partial\Omega \subset B_{R_0}$ and for $n > R_0$, we denote by $\Omega_n = \Omega \bigcap B_n$ and $\widehat{u}_{\Omega_n}$ the solution obtained in Step 1 with Ω replaced by

Ω_n. More precisely, for $n,\ k \in \mathbb{N}^*$, $\widehat{u}_{\Omega_n} = \lim_{k\to\infty} u_{n,k}$, where $u_{n,k}$ is the strong solution of the Cauchy problem:

$$\begin{cases} \frac{du_{n,k}}{dt}(t) + A_{\Omega_n}(u_{n,k}(t)) = 0,\ t \in (0,\infty), \\ u_{n,k}(0) = k \text{ in } \Omega_n. \end{cases} \tag{19}$$

It is clear that $u_{n,k} \le u_{n+1,k}$, therefore $\widehat{u}_{\Omega_n} \le \widehat{u}_{\Omega_{n+1}}$ in $\Omega_n \times (0,\infty)$. On the other hand, for every $x_0 \in \Omega$ there exists $n_0 \in \mathbb{N}$ such that $x_0 \in \Omega_n$ for all $n \ge n_0$. In view of the monotonicity of the sequence $\{\widehat{u}_{\Omega_n}(x_0,t)\}_{n\ge n_0}$ and (15) we can define $\widehat{u}_\Omega(x_0,t) = \lim_{n\to\infty} \widehat{u}_{\Omega_n}(x_0,t)\ \forall\ t > 0$.

Now, we are going to prove that $\widehat{u}_\Omega$ belongs to the space $\mathcal{S}_0(Q_\Omega^\infty)$ and is a maximal solution of problem $P_{\Omega,0}$. As we have been proved in Step 1, $\widehat{u}_{\Omega_n}$ satisfies

$$\frac{d\widehat{u}_{\Omega_n}}{dt}(t) + A_{\Omega_n}(\widehat{u}_{\Omega_n}(t)) = 0,\ t \in (0,\infty). \tag{20}$$

Taking the scalar product in $L^2(\Omega_n)$ of (20) and $\xi^2\widehat{u}_{\Omega_n}$ where $\xi \in C_0^\infty(\Omega_n)$, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_n} \xi^2\widehat{u}_{\Omega_n}(\cdot,t)^2dx + \int_{\Omega_n}\big(\mid \nabla\widehat{u}_{\Omega_n}\mid^2 + f(\widehat{u}_{\Omega_n})\widehat{u}_{\Omega_n}\big)(\cdot,t)\xi^2dx$$
$$+ 2\int_{\Omega_n}(\nabla\widehat{u}_{\Omega_n}(\cdot,t)\cdot\nabla\xi)\xi\widehat{u}_{\Omega_n}(\cdot,t)dx = 0\ \forall\ t > 0,$$

from which we derive

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_n} \xi^2\widehat{u}_{\Omega_n}(\cdot,t)^2dx + \int_{\Omega_n}\big(2^{-1}\mid \nabla\widehat{u}_{\Omega_n}\mid^2 + f(\widehat{u}_{\Omega_n})\widehat{u}_{\Omega_n}\big)(\cdot,t)\xi^2dx$$
$$\le 2\int_{\Omega_n}\xi^2 \mid\nabla\xi\mid^2 \widehat{u}_{\Omega_n}(\cdot,t)^2dx.$$

Let us consider $R > R_0,\ \tau > 0$. In what follows we denote by $C_k (k = 1,\cdots,4)$ different positive constants which depend on R and N, but are independent of n,t,τ. Taking ξ such that $0 \le \xi \le 1$, supp $\xi \subset B_{2R}^c$, $\xi \equiv 1$ on B_R, in the above inequality and integrating over $[\tau,t]$, we obtain:

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_R}\widehat{u}_{\Omega_n}(\cdot,t)^2dx + \int_\tau^t\int_{\Omega_R}\big(2^{-1}\mid\nabla\widehat{u}_{\Omega_n}\mid^2 + f(\widehat{u}_{\Omega_n})\big)\widehat{u}_{\Omega_n}dxds \\ &\quad\le 2\int_\tau^t\int_{\Omega_{2R}}\widehat{u}_{\Omega_n}^2 \mid\nabla\xi\mid^2 dxds + \frac{1}{2}\int_{\Omega_{2R}}\widehat{u}_{\Omega_n}(\cdot,\tau)^2dx \le C_1(t+1)\varphi(\tau)^2, \end{aligned} \tag{21}$$

$\forall\ n > 2R,\ t > \tau > 0$ (we have used inequality (15)). Letting $n \to \infty$ in (21) and using Fatou's lemma we find

$$\int_{\Omega_R}\widehat{u}_\Omega(\cdot,t)^2dx + \int_\tau^t\int_{\Omega_R}\big(\mid\nabla\widehat{u}_\Omega\mid^2 + 2f(\widehat{u}_\Omega)\widehat{u}_\Omega\big)dxds \le 2C_1(t+1)\varphi(\tau)^2, \tag{22}$$

for all $t > \tau$.

Next, if we take the scalar product in $L^2(\Omega_n)$ of (20) with $(t-\tau)\xi^2 d\widehat{u}_{\Omega_n}/dt$, we get

$$
\begin{aligned}
&(t-\tau)\int_{\Omega_n}\frac{d\widehat{u}_{\Omega_n}}{dt}(\cdot,t)^2\xi^2dx\\
&+\frac{d}{dt}\Big\{(t-\tau)\int_{\Omega_n}\big(2^{-1}\mid\nabla\widehat{u}_{\Omega_n}\mid^2+j(\widehat{u}_{\Omega_n})\big)(\cdot,t)\xi^2dx\Big\}\\
&-\int_{\Omega_n}\big(2^{-1}\mid\nabla\widehat{u}_{\Omega_n}\mid^2+j(\widehat{u}_{\Omega_n})\big)(\cdot,t)\xi^2dx\\
&\quad=-2(t-\tau)\int_{\Omega_n}\frac{d\widehat{u}_{\Omega_n}}{dt}(\cdot,t)(\nabla\widehat{u}_{\Omega_n}(\cdot,t)\cdot\nabla\xi)\xi dx\\
&\quad\le\frac{t-\tau}{2}\int_{\Omega_n}\frac{d\widehat{u}_{\Omega_n}}{dt}(\cdot,t)^2\xi^2dx+2(t-\tau)\int_{\Omega_n}\mid\nabla\widehat{u}_{\Omega_n}(\cdot,t)\mid^2\mid\nabla\xi\mid^2dx,
\end{aligned}
$$

for every $t>\tau$, where $j(x)=\int_0^x f(y)dy$ (see [3], Proposition 2, p. 517). Integrating this inequality over $[\tau,t]$ and taking ξ as before, we infer

$$
\begin{aligned}
\frac{1}{2}\int_\tau^t\int_{\Omega_R}(s-\tau)\frac{d\widehat{u}^2_{\Omega_n}}{dt}dxds&\\
+(t-\tau)\int_{\Omega_R}\big(2^{-1}\mid\nabla\widehat{u}_{\Omega_n}\mid^2+j(\widehat{u}_{\Omega_n})\big)(\cdot,t)dx&\\
\le C_2\int_\tau^t(s-\tau+1)\int_{\Omega_{2R}}\mid\nabla\widehat{u}_{\Omega_n}\mid^2dxds&\\
+C_3\int_\tau^t\int_{\Omega_{2R}}j(\widehat{u}_{\Omega_n})dxds,&
\end{aligned}
\tag{23}
$$

$\forall\ n>2R,\ t>\tau>0$. On account of inequalities (15) and (22), the right hand side of (23) remains bounded by $C_4(t+1)^2(\varphi(\tau)^2+j(\varphi(\tau)))$. Passing to the let as $n\to\infty$ in (23) we derive by Fatou's lemma:

$$
\begin{aligned}
\frac{1}{2}\int_\tau^t\int_{\Omega_R}(s-\tau)\frac{d\widehat{u}^2_{\Omega}}{dt}dxds&\\
+(t-\tau)\int_{\Omega_R}\big(2^{-1}\mid\nabla\widehat{u}_{\Omega}(\cdot,t)\mid^2+j(\widehat{u}_{\Omega}(\cdot,t))\big)dx&\\
\le C_4(t-\tau+1)^2(\varphi(\tau)^2+j(\varphi(\tau)))\ \forall\ t>\tau>0.&
\end{aligned}
\tag{24}
$$

Obviously, (23) is an estimate in $L^2(\tau,t;\widetilde{H}^1_0(\Omega_R))$, which is a closed subspace of $L^2(\tau,t;H^1(\Omega_R))$, therefore $\widehat{u}_\Omega\in L^2_{loc}(\tau,t;\widetilde{H}^1_0(\Omega_R))$. Note also that inequalities $(21),(23)$ imply $\widehat{u}_\Omega\in\mathcal{S}_0(Q^\infty_\Omega)$.

Finally, our next goal is to show that $\widehat{u}_\Omega$ is a maximal solution of problem $P_{\Omega,0}$. Let us consider $u\in\mathcal{S}_0(Q^\infty_\Omega)$. For every $\tau>0$ and $R\ge R_0$ we set $v_{R,\tau}(x,t):=u(x,t+\tau)-\widehat{u}_\Omega(x,t)-U_R(x)\ \forall\ (x,t)\in\Omega_R\times(0,\infty)$, where U_R is the large positive solution of equation (14) with $\Omega=B_R$. Obviously, supp $[v_{R,\tau}]_+(\cdot,t)$ is a compact subset of Ω_R $\forall\ t>0$. Also, by Lebesgue's theorem, $\lim_{s\searrow 0}\parallel [v_{R,\tau}]_+(\cdot,s)\parallel_{L^2(\Omega_R)}=0$, therefore making use of Remark 5, we get

$$u(x,t+\tau)\le U_R(x)+\widehat{u}_\Omega(x,t),$$

$\forall\ t>0$, for a.a. $x\in\Omega_R$. Letting $R\to\infty$, $\tau\searrow 0$, and using the continuity we find $u(x,t)\le\widehat{u}_\Omega(x,t)\ \forall\ t>0,\ x\in\Omega$ (we have used that $\lim_{R\to\infty}U_R(x)=0\ \forall\ x\in\Omega$, which is a consequence of Theorem 1). This concludes the proof.

Proposition 9. *Under assumptions of Theorem 8, the maximal solution $\widehat{u}_\Omega\in\mathcal{S}_0(Q_\Omega^\infty)$ is a minimal solution with initial blow-up, more exactly:*

$$\lim_{t\searrow 0}\widehat{u}_\Omega(x,t)=\infty \textit{ locally uniformly on } \Omega, \tag{25}$$

and $\forall\ u\in\mathcal{S}(Q_\Omega^\infty)$ *such that* $\lim_{t\searrow 0}u(x,t)=\infty$ *locally uniformly on* Ω *we have*

$$\widehat{u}_\Omega\le u \textit{ in } Q_\Omega^\infty. \tag{26}$$

In addition,

$$\lim_{t\searrow 0}\frac{\widehat{u}_\Omega(x,t)}{\varphi(t)}=1 \textit{ locally uniformly on } \Omega. \tag{27}$$

Proof. We will use the same notations as in the proof of Theorem 8. Let K be a compact subset of Ω. For $y\in K$, let $Q^\infty_{B_r(y)}:=B_r(y)\times(0,\infty)\subset Q_\Omega^\infty$ and denote by $U_{r,y}$ the unique positive large solution of the problem

$$\begin{cases} -\Delta U_{r,y}+f(U_{r,y})=0 \text{ in } B_r(y),\\ \lim\limits_{|x-y|\to r}U_{r,y}(x)=\infty. \end{cases}$$

For $\tau>0$, there exists $k_\tau\in\mathbb{N}$, such that $u_{k_\tau}(x,0)=k_\tau>\varphi(\tau)$. We denote by $v_{\tau,r}$ the function defined as follows

$$v_{\tau,r}:=\varphi(t+\tau)-\widehat{u}_\Omega(x,t)-U_{r,y}(x)\ \forall (x,t)\in Q^\infty_{B_r(y)}.$$

Obviously,

$$\widehat{u}_\Omega(x,t)+U_{r,y}(x)\ge u_{k_\tau}(x,t)+U_{r,y}(x)>\varphi(t+\tau)\ \forall\ t>0,\quad for\ a.a.\ x\in\partial B_r(y),$$

therefore, for all $t>0$, supp $[v_{\tau,r}]_+(\cdot,t)$ is a compact subset of $B_r(y)$. In addition, $\lim_{t\searrow 0}\parallel [v_{\tau,r}]_+(\cdot,t)\parallel=0\ \forall\ t>0$ which implies (see Remark 5) $\widehat{u}_\Omega(x,t)+U_{r,y}(x)\ge\varphi(t+\tau)\ \forall\ t>0$ for a.a. $x\in B_r(y)$. Therefore, using the continuity and letting $\tau\to 0$ in the previous inequality we infer

$$\widehat{u}_\Omega(x,t)+U_{r,y}(x)\ge\varphi(t)\ \forall\ (x,t)\in Q^\infty_{B_r(y)}. \tag{28}$$

On the other hand, for any $0<r'<r$ there exists a positive constant $C_{y,r'}$ such that $U_{r,y}(x)\le C_{y,r'}\ \forall\ x\in B_{r'}(y)$. This inequality together with (28) implies $\lim\limits_{t\searrow 0}\widehat{u}_\Omega(x,t)=\infty$ uniformly on $B_{r'}(y)$. As K can be covered by a finite number of such balls we obtain (25).

Next we are going to prove (27). Making use of inequalities (15) and (28) (with Ω replaced by $B_r(y)$) we obtain

$$\widehat{u}_{B_r(y)}(x,t) + U_{r,y}(x) \geq \varphi(t) \geq \widehat{u}_{B_r(y)}(x,t) \ \forall \ (x,t) \in Q^{\infty}_{B_r(y)},$$

and hence

$$\lim_{t\searrow 0} \frac{\widehat{u}_{B_r(y)}(x,t)}{\varphi(t)} = 1 \text{ locally uniformly on } B_{r'}(y) \ \forall \ r' < r. \tag{29}$$

Since

$$\widehat{u}_{B_r(y)}(x,t) + U_{r,y}(x) \geq \widehat{u}_{\Omega}(x,t) \geq \widehat{u}_{B_r(y)}(x,t) \ \forall \ (x,t) \in Q^{\infty}_{B_r(y)},$$

the last equality (29) leads us to (27).

Finally, if $u \in \mathcal{S}(Q^{\infty}_{\Omega})$ is a solution of (1) which satisfies initial blow-up condition locally uniformly on Ω, and Ω is a bounded subdomain of $\mathbb{R}^N$ we have $u_k \leq u$ in Q^{∞}_{Ω} and letting $k \to \infty$ we derive (26). On the other hand, when Ω is unbounded we have $\widehat{u}_{\Omega_n} \leq u$ in $Q^{\infty}_{\Omega_n}$, where $\Omega_n = \Omega \bigcap B_n$. Letting $n \to \infty$ in the previous inequality, (26) follows. The proof is complete.

Remark 10. In fact, (27) remains true if we replaced $\widehat{u}_{\Omega}$ by any solution $u \in \mathcal{S}(Q^{\infty}_{\Omega})$ which verifies the initial blow-up condition on Ω. The proof follows ideas similar to those we have already used in the proof of Proposition 9.

If $\partial\Omega$ has the minimal regularity which allows the Dirichlet problem to be solved by any continuous function g given on $\partial_l Q^{\infty}_{\Omega}$, we can consider another construction of the maximal solution of problem $P_{\Omega,0}$. The needed assumption on $\partial\Omega$ is known as the parabolic Wiener criterion (PWC) (see [11]).

Definition 11. *If $\partial\Omega$ is compact and satisfies PWC, we denote by $\mathbf{S}_0(Q^{\infty}_{\Omega})$ the set of all positive solutions of problem $P_{\Omega,0}$ which belong to the space $C^{2,1}(Q^{\infty}_{\Omega})$ $\bigcap C(\overline{\Omega} \times (0,\infty))$.*

Theorem 12. *Assume that $(f_1)-(f_3)$ are satisfied, $\partial\Omega$ is compact and satisfies PWC. Then problem $P_{\Omega,0}$ has a maximal solution $\overline{u}_{\Omega} \in \mathbf{S}_0(Q^{\infty}_{\Omega})$.*

Proof. We first assume that Ω is bounded. In the same manner as in the proof of Theorem 8 we can construct the maximal solution $\overline{u}_{\Omega}$ as being the limit in $\overline{\Omega} \times (0,\infty)$ of the increasing sequence $\{u_k\}_{k\geq 1}$, where u_k is the unique solution (on account of the maximum principle) of the problem $P_{\Omega,0}$, which satisfies the initial condition $u_k(\cdot, 0) = k$ in Ω for all $k \geq 1$. Clearly, such a solution u_k exists and belongs to $\mathbf{S}_0(Q^{\infty}_{\Omega})$ on account of PWC assumption. Furthermore, inequality (14) is still valid for u_k in $\overline{\Omega} \times (0,\infty)$. In view of this estimate, the sequence $\{u_k\}_{k\geq 1}$, is locally uniformly bounded in $\overline{\Omega} \times (0,\infty)$ and therefore (by standard parabolic equations estimates) it is compact in $C^{2,1}(Q^{\infty}_{\Omega})$.

Consequently, $\overline{u}_{\Omega} := \lim_{k\to\infty} u_k$ belongs to $C^{2,1}(Q^{\infty}_{\Omega}) \bigcap C(\overline{\Omega} \times (0,\infty))$ and verifies (1) in Q^{∞}_{Ω}. In addition, taking into account the construction of $\overline{u}_{\Omega}$ we get $\overline{u}_{\Omega} = 0$ on $\partial\Omega \times (0,\infty)$, therefore $u \in \mathbf{S}_0(Q^{\infty}_{\Omega})$.

On the other hand, if Ω is unbounded we consider $R_0 > 0$ such that $\partial\Omega \subset B_{R_0}$ and for $n > R_0,\ n \in \mathbb{N}$ we denote by $\overline{u}_{\Omega_n}$ the solution obtained above with Ω replaced by $\Omega_n = \Omega \bigcap B_n$. From the construction of $\overline{u}_{\Omega_n}$ we derive that $\overline{u}_{\Omega_n} \leq \overline{u}_{\Omega_{n+1}}$ in $Q^{\infty}_{\Omega_n}$. Since $\overline{u}_{\Omega_n} \leq \varphi$ in $\overline{\Omega}_n \times (\tau, \infty)\ \forall\ \tau > 0$ we can see that there exists $\overline{u}_{\Omega} := \lim_{n\to\infty} \overline{u}_{\Omega_n}$ in Q^{∞}_{Ω}. By applying a bootstrap regularity argument we obtain $\overline{u}_{\Omega} \in \mathbf{S}_0(Q^{\infty}_{\Omega})$.

In order to establish the maximality of $\overline{u}_{\Omega}$ in $\mathbf{S}_0(Q^{\infty}_{\Omega})$ we can use similar reasonings to those we have already used in the last part of the proof of Theorem 8, so we will just sketch its. For example, let us suppose that Ω is unbounded. We will use the same notation as in the proof of the previous Theorem. If $u \in \mathbf{S}_0(Q^{\infty}_{\Omega})$, for $R > R_0,\ \tau > 0$ we set

$$v_{R,\tau} := u(\cdot, \cdot + \tau) - \overline{u}_{\Omega} - U_R \text{ in } \Omega_R \times (0, \infty).$$

Clearly, $[v_{R,\tau}]_+$ is a subsolution of (1) in $\Omega_R \times (0, \infty)$ which is zero on $(\partial\Omega_R \times (0, \infty)) \bigcup (\Omega_R \times \{0\}.)$ As a consequence of the maximum principle we get $v_{R,\tau} \leq 0$ in $\Omega_R \times (0, \infty)$. Letting $R \to \infty,\ \tau \to 0$ in the last inequality we obtain $u \leq \overline{u}_{\Omega}$ in Q^{∞}_{Ω}.

Proposition 13. *Under assumptions of Theorem 12*

$$\overline{u}_{\Omega} \in \mathcal{S}_0(Q^{\infty}_{\Omega}),\ \overline{u}_{\Omega} = \widehat{u}_{\Omega}. \tag{30}$$

Proof. Let us suppose that Ω is bounded. As $\partial\Omega$ verifies PWC, there exists a sequence $\{\Omega^n\}_n$ of smooth domains such that

$$\Omega^n \subset \overline{\Omega}^n \subset \Omega^{n+1} \subset \Omega,\ \bigcup_n \Omega^n = \Omega,\ \ \sup\{\text{dist } (x, \Omega^c)\ ; x \in \partial\Omega^n\} < 1/n.$$

For $\tau > 0,\ n \in \mathbb{N}$, we consider the solutions $u_{n,\tau} \in C^{2,1}(\Omega^n \times (\tau, \infty)) \bigcap C(\overline{\Omega}^n \times (\tau, \infty))$ of the following problems:

$$\begin{cases} \partial_t u_{n,\tau} - \Delta u_{n,\tau} + f(u_{n,\tau}) = 0 \ \text{ in } \ \Omega^n \times (\tau, \infty), \\ u_{n,\tau} = 0 \ \text{ on } \ \partial\Omega^n \times [\tau, \infty), \\ u_{n,\tau}(\cdot, \tau) = \overline{u}_{\Omega}(\cdot, \tau) \ \text{ in } \Omega^n. \end{cases} \tag{31}$$

By the maximum principle, $\forall\ t > \tau$ we have

$$0 \leq \overline{u}_{\Omega}(\cdot, t) - u_{n,\tau}(\cdot, t) \leq \max\{\overline{u}_{\Omega}(x, s)\ ;\ (x, s) \in \partial\Omega^n \times [\tau, t]\} \to 0 \text{ as } n \to \infty.$$

This implies that

$$\lim_{n\to\infty} \widetilde{u}_{n,\tau} = \overline{u}_{\Omega} \text{ uniformly on } \overline{\Omega} \times [\tau, t]\ \forall\ t > \tau > 0, \tag{32}$$

where $\widetilde{u}_{n,\tau}$ is the extension by 0 outside $\Omega^n \times [\tau, \infty)$. As $\partial\Omega^n$ is smooth we obtain $u_{n,\tau}(\cdot, t) \in H^1_0(\Omega^n)$ therefore $\widetilde{u}_{n,\tau}(\cdot, t) \in H^1_0(\Omega)\ \forall\ t > \tau$. A standard computation involving Green's formulae and inequality (15) leads us to the estimates

$$\frac{1}{2}\int_{\Omega} \widetilde{u}_{n,\tau}(\cdot, t)^2 dx + \int_{\tau}^{t}\int_{\Omega} \big(\mid \nabla \widetilde{u}_{n,\tau} \mid^2 + f(\widetilde{u}_{n,\tau})\widetilde{u}_{n,\tau}\big) dx ds \leq C_1 \varphi(\tau)^2, \tag{33}$$

$$\int_\tau^t \int_\Omega (s-\tau)\frac{d\widetilde{u}_{n,\tau}}{ds}^2 dxds + (t-\tau)\int_\Omega \big(2^{-1} \mid \nabla \widetilde{u}_{n,\tau} \mid^2 + j(\widetilde{u}_{n,\tau})\big)(\cdot,t)dx \\ \leq C_2(t-\tau+1)^2(\varphi^2(\tau)+j(\varphi(\tau)))\ \forall\ t>\tau>0, \tag{34}$$

where C_1, C_2 are positive constants depending on Ω. We let $n \to \infty$ and derive by Fatou's lemma and (32) that estimates (33) and (34) still hold with $\overline{u}_\Omega$ instead of $\widetilde{u}_{n,\tau}$. Obviously these inequalities imply $\overline{u}_\Omega \in \mathcal{S}_0(Q_\Omega^\infty)$.

If Ω is unbounded we can consider smooth internal approximations of bounded domains, $\{\Omega_n^m\}_m$ of $\Omega_n := \Omega \bigcap B_n\ \forall\, n > R_0$, as above. Therefore $\bigcup_{m,n} \Omega_n^m = \Omega$. For $\tau > 0$, we denote by $u_{m,n,\tau}$ the solutions of problems (31) with Ω^n replaced by Ω_n^m. Using the same arguments as in the case when Ω is bounded we derive

$$\lim_{m,n\to\infty} \widetilde{u}_{m,n,\tau} = \overline{u}_\Omega \text{ uniformly on } \overline{\Omega}\times[\tau,t]\ \forall\ t>\tau>0.$$

Next, reasoning as in the proof of Theorem 12 we can see that inequalities (22) and (24) hold with $\widehat{u}$ replaced by $\overline{u}_\Omega$, therefore $\overline{u}_\Omega \in \mathcal{S}_0(Q_\Omega^\infty)$.

Finally, by the construction, $\overline{u}_\Omega$ is a solution which satisfies initial blow-up condition locally uniformly on Ω and belongs to $\mathcal{S}_0(Q_\Omega^\infty)$. In conclusion, taking into account Theorem 8 and Proposition 9 we obtain $\overline{u}_\Omega = \widehat{u}_\Omega$. The proof is complete.

3. Uniqueness of initial blow-up solutions

As a consequence of the results obtained in Section 2 we are able to state and prove the main Theorem of this paper concerning the existence and uniqueness for the solution of problem $P_{\Omega,g}$, consisting of equation (1) and boundary condition (2), which verifies initial blow-up condition:

Theorem 14. *Let Ω be a domain in $\mathbb{R}^N$ with a compact boundary satisfying PWC. Assume that f verifies assumptions $(f_1)-(f_3)$. Then for any $g \in C(\partial_l Q_\Omega^\infty)$, $g \geq 0$ there exists a unique positive solution $\overline{u}_{\Omega,g} \in C^{2,1}(Q_\Omega^\infty)\bigcap C(\overline{\Omega}\times(0,\infty))$ of the following problem*

$$\begin{cases} u_t - \Delta u + f(u) = 0 \quad in \quad Q_\Omega^\infty, \\ u = g \quad on \quad \partial_l Q_\Omega^\infty, \\ \lim\limits_{t\searrow 0} \overline{u}_{\Omega,g}(x,t) = \infty \text{ locally uniformly on } \Omega. \end{cases} \tag{35}$$

Proof. Step 1. Existence.

The construction of a solution verifying (34) is similar to that we have already used in the proof of Theorem 12. Thus, we are not going into many details but we will just describe the basic steps to be followed when, for example, Ω is bounded. More exactly, for $\tau > 0$, $k \in \mathbb{N}$, we denote by $u_{k,\tau,g}$ the unique positive solution belongs to $C^{2,1}(Q_\Omega^{\tau,\infty})\bigcap C(\overline{\Omega}\times(0,\infty))$, $Q_\Omega^{\tau,\infty} := \Omega\times(\tau,\infty)$, of the problem

$$\begin{cases} u_t - \Delta u + f(u) = 0 \text{ in } Q_\Omega^{\tau,\infty}, \\ u = g \text{ on } \partial\Omega\times(\tau,\infty), \\ u(\cdot,\tau) = k \text{ in } \Omega. \end{cases}$$

In view of the following estimate

$$u_{k,\tau,g}(x,t) \le \widehat{u}_\Omega(x,t-\tau) + u_{0,\tau,g}(x,t)\ \forall\ (x,t) \in Q_{\overline{\Omega}}^{\tau,\infty} := \overline{\Omega} \times (\tau,\infty)\ \forall\ k \in \mathbb{N},$$

the increasing sequence $\{u_{k,\tau,g}\}_k$ is locally uniformly bounded in $Q_{\overline{\Omega}}^{\tau,\infty}$. Therefore, there exists $\lim_{k\to\infty} u_{k,\tau,g}(x,t) := u_{\infty,\tau,g}(x,t)$ for all $(x,t) \in Q_{\overline{\Omega}}^{\tau,\infty}$. Obviously, $u_{\infty,\tau,g} \in C(Q_{\overline{\Omega}}^{\tau,\infty})$, $u_{\infty,\tau,g} = g$ on $\partial\Omega\times(\tau,\infty)$ and by the parabolic regularity theory we obtain that $u_{\infty,\tau,g}$ belongs to $C^{2,1}(Q_\Omega^{\tau,\infty})$ and verifies (1) in $Q_\Omega^{\tau,\infty}$. Clearly, $\lim_{t\searrow\tau} u_{\infty,\tau,g}(x,t) = \infty$ locally uniformly on Ω. Since, $u_{\infty,\tau,g} \ge u_{\infty,\tau',g}$ in $Q_{\overline{\Omega}}^{\tau}$ for all $0 < \tau' < \tau$ there exists $\overline{u}_{\Omega,g}(x,t) = \lim_{t\searrow\tau} u_{\infty,\tau,g}(x,t)$ for all $(x,t) \in \overline{\Omega} \times (0,\infty)$. Obviously, $\widetilde{u}_{\Omega,g}$ is a solution of problem (35).

If Ω is unbounded we define $\overline{u}_{\Omega,g}(x,t) = \lim_{n\to\infty} \overline{u}_{\Omega_n,g}(x,t)$ for all $(x,t) \in \overline{\Omega} \times (0,\infty)$, where $\overline{u}_{\Omega_n,g}$ is the solution of problem (35) with Ω replaced by $\Omega_n = \Omega \bigcap B_n$.

Step 2. Uniqueness.

Assume that there exists another positive solution $u_g \in C(\overline{\Omega}\times(0,\infty)) \bigcap C^{2,1}(Q_\Omega^\infty)$ of problem (35) and we will verify that $u_g = \overline{u}_{\Omega,g}$.

As $u_{\infty,\tau,g}$ dominates in $\Omega \times (\tau,\infty)$ the restriction to this set of any solution $u \in C^{2,1}(Q_\Omega^\infty) \bigcap C(\overline{\Omega} \times (0,\infty))$ of problem $P_{\Omega,g}$, we have $u_{\infty,\tau,g} \ge u_g$ in $Q_\Omega^{\tau,\infty}$ and letting $\tau \searrow 0$ we get

$$u_g \le \overline{u}_{\Omega,g} \text{ in } Q_\Omega^\infty. \tag{36}$$

Now, for every $\tau > 0$ we denote by u_τ the solution of the problem

$$\begin{cases} u_t - \Delta u + f(u) = 0 \ \text{ in } \ Q_\Omega^{\tau,\infty}, \\ u = 0 \ \text{ on } \ \partial\Omega \times (\tau,\infty), \\ u(\cdot,\tau) = u_g(\cdot,\tau) \ \text{ in } \Omega. \end{cases} \tag{37}$$

Similarly, we denote by $\overline{u}_\tau$ the unique positive solution of the problem $(37)_{1,2}$ with initial condition at $t = \tau,\ u(\cdot,\tau) = \overline{u}_{\Omega,g}(\cdot,\tau)$ in Ω. We have $u_\tau \le u_g,\ u_\tau \le \overline{u}_\tau \le \overline{u}_{\Omega,g}$ in $Q_\Omega^{\tau,\infty}$. Put

$$W_g := \overline{u}_{\Omega,g} - u_g,\ \lambda := p(\overline{u}_{\Omega,g}, u_g) \text{ in } Q_\Omega^\infty,\ W_{\tau,g} := \overline{u}_\tau - u_\tau,\ \lambda_\tau := p(\overline{u}_\tau, u_\tau) \text{ in } Q_\Omega^\tau,$$

where

$$p(r,s) := \begin{cases} \frac{f(r)-f(s)}{r-s} & \text{if } \ r \ne s, \\ 0 & \text{if } r = s. \end{cases}$$

Observe that the convexity of function f implies $p(r_2,s_2) \ge p(r_1,s_1)\ \forall\ r_1 \ge s_1,\ r_2 \ge s_2,\ r_2 \ge r_1,\ s_2 \ge s_1$. Therefore $\lambda \ge \lambda_\tau$ in Q_Ω^τ and a straightforward calculation yields

$$\partial_t(W_g - W_{\tau,g}) - \triangle(W_g - W_{\tau,g}) + \lambda(W_g - W_{\tau,g}) \le 0 \text{ in } Q_\Omega^{\tau,\infty}. \tag{38}$$

Furthermore, $W_g - W_{\tau,g} = 0$ on $\big(\partial\Omega \times [\tau,\infty)\big) \bigcup \big(\Omega \times \{\tau\}\big)$. Thus, according to the maximum principle we obtain that

$$W_g \le W_{\tau,g} \text{ in } Q_\Omega^{\tau,\infty}. \tag{39}$$

On the other hand,

$$u_\tau(x,\tau) = u_g(x,\tau) \geq u_{\tau'}(x,\tau),\ \overline{u}_\tau(x,\tau) = \overline{u}_{\Omega,g}(x,\tau) \geq \overline{u}_{\tau'}(x,\tau),$$

for all $x \in \Omega, 0 < \tau' < \tau$, therefore there exist the limits $\lim_{\tau\searrow 0} u_\tau(x,t) := u_0(x,t)$ and $\lim_{\tau\searrow 0} \overline{u}_\tau(x,t) := \overline{u}_0(x,t)$ for all $(x,t) \in Q_\Omega^\infty$. In addition, by standard arguments we have already used it follows that u_0 and $\overline{u}_0$ are solutions of problem (35) with $g = 0$ which belong to $C^{2,1}(Q_\Omega^\infty)\bigcap C(\overline{\Omega}\times(0,\infty))$. If we take into account (39) and let $\tau \searrow 0$, we infer

$$\overline{u}_{\Omega,g} - u_g \leq \widetilde{u}_0 - u_0 \text{ in } Q_\Omega^\infty. \tag{40}$$

Since $\overline{u}_{\Omega,g} \geq \overline{u}_\Omega$ we can derive from maximum principle that $\overline{u}_0 \geq \overline{u}_\Omega$, which implies $\overline{u}_0 = \overline{u}_\Omega$ (we have used the maximality of $\overline{u}_\Omega$). On the other hand, on account of Proposition 9 and Theorem 12, $u_0 \geq \widehat{u}_\Omega = \overline{u}_\Omega$. Therefore, the right-hand side of inequality (40) is negative. This together with (36) leads to $\overline{u}_{\Omega,g} = u_g$ in Q_Ω^∞. The proof is complete.

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