

On homoclinic solutions of a semilinear p -Laplacian difference equation with periodic coefficients

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Abstract

We study the existence of homoclinic solutions for semilinear p -Laplacian difference equations with periodic coefficients. The proof of the main result is based on Brezis–Nirenberg’s Mountain Pass Theorem. Several examples and remarks are given.

Key words: homoclinic orbits, eigenvalue problem, difference equations, mountain-pass theorem.

*Dedicated to Professor Gheorghe Moroşanu
on the occasion of his 60-th birthday*

1 Introduction

This paper is concerned with the study of the existence of homoclinic solutions for the p -Laplacian difference equation

$$\begin{aligned}\Delta_p^2 u(k-1) - V(k) u(k) |u(k)|^{q-2} + \lambda f(k, u(k)) &= 0, \\ u(t) \rightarrow 0, |t| \rightarrow \infty,\end{aligned}$$

where $u(k)$, $k \in \mathbb{Z}$ is a sequence of real numbers, Δ is the difference operator $\Delta u(k) = u(k+1) - u(k)$,

$$\Delta_p^2 u(k-1) = \Delta u(k) |\Delta u(k)|^{p-2} - \Delta u(k-1) |\Delta u(k-1)|^{p-2}$$

is referred as the p -Laplacian difference operator and functions $V(k)$ and $f(k, x)$ are T -periodic in k and satisfy suitable conditions.

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In the theory of differential equations, a trajectory $x(t)$, which is asymptotic to a constant as $|t| \rightarrow \infty$ is called doubly asymptotic or homoclinic orbit. The notion of homoclinic orbit is introduced by Poincaré [17] for continuous Hamiltonian systems.

Recently, there is a large literature on the use of variational methods to the existence of homoclinic or heteroclinic orbits of Hamiltonian systems; see [5], [9], [16], [18], [19], [20] and the references therein.

In the recent paper of Li [12] a unified approach to the existence of homoclinic orbits for some classes of ODE's with periodic potentials is presented. It is based on the Brezis and Nirenberg's mountain-pass theorem [2]. In this paper we extend this approach to homoclinic orbits for discrete p -Laplacian type equations.

Discrete boundary value problems have been intensively studied in the last decade. The studies of such kind of problems can be placed at the interface of certain mathematical fields, such as nonlinear differential equations and numerical analysis. On the other hand, they are strongly motivated by their applicability to mathematical physics and biology.

The variational approach to the study of various problems for difference equations has been recently applied in, among others, the papers of Agarwal, Perera and O'Regan [1], Cabada, Iannizzotto and Tersian [3], Chen and Fang [4], Fang and Zhao [7], Jiang and Zhou [10], Ma and Guo [13], Mihăilescu, Rădulescu and Tersian [15], Kristály, Mihăilescu, Rădulescu and Tersian [11].

Along the paper, given two integer numbers $a < b$, we shall denote $[a, b] = \{a, \dots, b\}$. Moreover, for every $p > 1$, we consider the following function

$$\varphi_p(t) = t|t|^{p-2}, \Phi_p(t) = \frac{|t|^p}{p}.$$

It is obvious that $\Phi_p'(t) = \varphi_p(t)$ for all $t \in \mathbb{R}$ and $p \neq 0$. Moreover

$$\Delta_p^2 u(k-1) = \Delta(\varphi_p(\Delta u(k-1))).$$

Suppose that

$$V : \mathbb{Z} \rightarrow \mathbb{R} \text{ is a } T\text{-periodic positive potential} \quad (1)$$

and

$$0 < V_0 = \min \{V(0), \dots, V(T-1)\} \leq \max \{V(0), \dots, V(T-1)\} = V_1. \quad (2)$$

Denote

$$A(u) = \sum_{k \in \mathbb{Z}} \Phi_p(\Delta u(k-1)) + \sum_{k \in \mathbb{Z}} V(k) \Phi_q(u(k)). \quad (3)$$

Let us consider functions f satisfying the following assumptions:

$$(F_1) \quad \text{The function } f(k, t) \text{ is continuous in } t \in \mathbb{R} \text{ and } T\text{-periodic in } k.$$

The potential function $F(k, t)$ of $f(k, t)$

$$F(k, t) = \int_0^t f(k, s) ds \quad (4)$$

satisfies the Rabinowitz's type condition:

(F₂) There exist $\mu > p \geq q > 1$ and $s > 0$ such that

$$\begin{aligned} \mu F(k, t) &\leq t f(k, t), \quad k \in \mathbb{Z}, \quad t \neq 0 \\ \text{and } F(k, t) &> 0, \quad \forall k \in \mathbb{Z}, \quad \text{for } t \geq s > 0. \end{aligned}$$

(F₃) $f(k, t) = o(|t|^{q-1})$ as $|t| \rightarrow 0$.

Further we consider the semilinear eigenvalue p -Laplacian difference equation

$$\Delta_p^2 u(k-1) - V(k) u(k) |u(k)|^{q-2} + \lambda f(k, u(k)) = 0, \quad (5)$$

where $\lambda > 0$ and we are looking for its homoclinic solutions, i.e., solutions of Eq. (5) such that $u(k) \rightarrow 0$ as $|k| \rightarrow \infty$.

In order to obtain homoclinic solutions of Eq. (5), we will use variational approach and Brezis-Nirenberg mountain pass theorem [2].

To this end, consider the functional $J : \ell^q \rightarrow \mathbb{R}$, defined as

$$J(u) = A(u) - \lambda \sum_{k \in \mathbb{Z}} F(k, u(k)).$$

Our main result is the following

Theorem 1 *Suppose that the function $V : \mathbb{Z} \rightarrow \mathbb{R}$ is positive and T -periodic and the functions $f(k, \cdot) : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy assumptions (F₁)-(F₃). Then, for each $\lambda > 0$, the equation (5) has a nonzero homoclinic solution $u \in \ell^q$, which is a critical point of the functional $J : \ell^q \rightarrow \mathbb{R}$.*

Moreover, given a nontrivial solution u of problem (5), there exist k_{\pm} two integer numbers such that for all $k > k_+$ and $k < k_-$, the sequence $u(k)$ is strictly monotone.

The paper is organized as follows. In Section 2, we present the proof of the main result and discuss the optimality of the condition (F₂). In Section 3, we give some examples of equations modeled by this kind of problems and present some additional remarks.

2 Proof of the main result

Let $u = \{u(k) : k \in \mathbb{Z}\}$ be a sequence, $q > 1$ and

$$\begin{aligned} \ell^q &= \left\{ u : |u|_q^q = \sum_{k \in \mathbb{Z}} |u(k)|^q < \infty \right\}, \\ \ell^\infty &= \left\{ u : |u|_\infty = \sup_{k \in \mathbb{Z}} |u(k)| < \infty \right\}. \end{aligned}$$

It is well known that if $0 < q \leq p$, then $\ell^q \subseteq \ell^p$. Indeed, if $\sum_{k \in \mathbb{Z}} |u(k)|^q < \infty$, there exists a positive integer number R , such that for all k satisfying $|k| > R$ it is verified that $|u(k)|^q < 1$ and, as consequence, $|u(k)|^p \leq |u(k)|^q$ and the series $\sum_{k \in \mathbb{Z}} |u(k)|^p$ is convergent too.

Consider now the functional $J : \ell^q \rightarrow \mathbb{R}$, defined as

$$J(u) = A(u) - \lambda \sum_{k \in \mathbb{Z}} F(k, u(k)), \quad (6)$$

with A given in (3) and F defined in (4).

We have the following result

Lemma 2 *The functional $J : \ell^q \rightarrow \mathbb{R}$ is well defined, C^1 -differentiable and its critical points are solutions of Eq. (5).*

Proof. By using the inequality for nonnegative a and b and $p > 1$

$$\left(\frac{a+b}{2}\right)^p \leq \frac{a^p + b^p}{2},$$

and the inclusion $\ell^q \subseteq \ell^p$ for $1 < q \leq p$, it follows that

$$\sum_{k \in \mathbb{Z}} |\Delta u(k-1)|^p \leq 2^{p-1} \sum_{k \in \mathbb{Z}} (|u(k)|^p + |u(k-1)|^p) = 2^p \sum_{k \in \mathbb{Z}} |u(k)|^p < \infty.$$

Now, let's see that the series $\sum_{k \in \mathbb{Z}} F(k, u(k))$ is convergent: by using (F_3) , it follows that there exist $\delta \in (0, 1)$ and sufficiently large N such that

$$F(k, u(k)) < |u(k)|^q \quad \text{for } |u(k)|^q < \delta < 1 \quad \text{and } |k| > N.$$

Then, the series $\sum_{k \in \mathbb{Z}} F(k, u(k))$ is convergent and the functional J is well-defined on ℓ^q .

It is Gâteaux differentiable and for $v \in \ell^q$:

$$\begin{aligned} \langle J'(u), v \rangle &= \lim_{t \rightarrow 0} \frac{J(u+tv) - J(u)}{t} \\ &= \sum_{k \in \mathbb{Z}} \Delta u(k-1) |\Delta u(k-1)|^{p-2} \Delta v(k-1) \\ &\quad + \sum_{k \in \mathbb{Z}} V(k) u(k) |u(k)|^{q-2} v(k) - \lambda \sum_{k \in \mathbb{Z}} f(k, u(k)) v(k) \end{aligned}$$

and partial derivatives

$$\frac{\partial J(u)}{\partial u(k)} = -\Delta_p^2 u(k-1) + V(k) u(k) |u(k)|^{q-2} - \lambda f(k, u(k)), \quad (7)$$

are continuous functions.

Moreover the functional J is continuously Fréchet-differentiable in ℓ^q . It is clear, by (7), that the critical points of J are solutions of Eq. (5). \square

To obtain homoclinic solutions of Eq. (5) we will use mountain-pass theorem of Brezis and Nirenberg [2]. Recall its statement. Let X be a Banach space with norm $\|\cdot\|$, and $I : X \rightarrow \mathbb{R}$ be a C^1 -functional. I satisfies the $(PS)_c$ condition if every sequence (x_k) of X such that

$$I(x_k) \rightarrow c \text{ and } I'(x_k) \rightarrow 0, \quad (8)$$

has a convergent subsequence. A sequence $(x_k) \subset X$ such that (8) holds is referred as $(PS)_c$ -sequence.

Theorem 3 (Mountain-pass theorem, Brezis & Nirenberg [2]) *Let X be a Banach space with norm $\|\cdot\|$, $I \in C^1(X, \mathbb{R})$ and suppose that there exist $r > 0$, $\alpha > 0$ and $e \in X$ such that $\|e\| > r$*

- (i) $I(x) \geq \alpha$ if $\|x\| = r$,
- (ii) $I(e) < 0$.

Let $c = \inf_{\gamma \in \Gamma} \left\{ \max_{0 \leq t \leq 1} I(\gamma(t)) \right\} \geq \alpha$, where

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

Then, there exists a $(PS)_c$ sequence for I . Moreover, if I satisfies the $(PS)_c$ condition, then c is a critical value of I , i.e., there exists $u_0 \in X$ such that $I(u_0) = c$ and $I'(u_0) = 0$.

Note that, by assumption (1), the norm $|\cdot|_q$ in ℓ^q is equivalent to

$$\|u\|_q^q = \frac{1}{q} \sum_{k \in \mathbb{Z}} V(k) |u(k)|^q. \quad (9)$$

Lemma 4 Suppose that (F_1) - (F_3) hold, then there exist $\rho > 0$, $\alpha > 0$ and $e \in \ell^q$ such that $\|e\|_q > \rho$ and

- (1) $J(u) \geq \alpha$ if $\|u\|_q = \rho$,
- (2) $J(e) < 0$.

Proof. By (F_3) , there exists $\delta \in (0, 1)$ such that

$$F(k, t) \leq \frac{V_0}{2q\lambda} |t|^q \text{ if } |t| \leq \delta. \quad (10)$$

Let $\rho = \left(\frac{V_0}{q}\right)^{1/q} \delta$ (V_0 defined in (2)), then, for u , $\|u\|_q = \rho$,

$$\begin{aligned} \frac{V_0}{q} \delta^q &= \rho^q = \|u\|_q^q = \frac{1}{q} \sum_{k \in \mathbb{Z}} V(k) |u(k)|^q \\ &\geq \frac{V_0}{q} |u(k)|^q \text{ for all } k \in \mathbb{Z}, \end{aligned}$$

which implies that $|u(k)| \leq \delta$ for all $k \in \mathbb{Z}$.

Hence, by (10)

$$\begin{aligned} \sum_{k \in \mathbb{Z}} F(k, u(k)) &\leq \frac{V_0}{2q\lambda} \sum_{k \in \mathbb{Z}} |u(k)|^q \\ &\leq \frac{1}{2q\lambda} \sum_{k \in \mathbb{Z}} V(k) |u(k)|^q = \frac{1}{2\lambda} \|u\|_q^q \end{aligned}$$

and

$$\begin{aligned} J(u) &= A(u) - \lambda \sum_{k \in \mathbb{Z}} F(k, u(k)) \\ &\geq \|u\|_q^q - \frac{1}{2} \|u\|_q^q = \frac{1}{2} \|u\|_q^q = \frac{\rho^q}{2} > 0. \end{aligned}$$

By (F_2) , there exist $c_1, c_2 > 0$ such that $F(k, t) \geq c_1 t^\mu - c_2$ for all $t > 0$ and $k \in \mathbb{Z}$.

Take $v \in \ell^q$, $v(0) = a > 0$, $v(k) = 0$ if $k \neq 0$. Then, since $\mu > p \geq q$

$$\begin{aligned} J(\kappa v) &= A(\kappa v) - \lambda \sum_{k \in \mathbb{Z}} F(k, \kappa v(k)) \\ &\leq \frac{2}{p} \kappa^p a^p + V(0) \frac{\kappa^q a^q}{q} - \lambda(c_1 \kappa^\mu a^\mu - c_2) \\ &< 0, \end{aligned} \tag{11}$$

if κ is sufficiently large.

Then, we can take κ large enough, such that for $e = \kappa v$, $\|e\|_q^q = V(0) \frac{\kappa^q a^q}{q} > \rho^q$ and (11) holds. \square

Lemma 5 *Suppose that the assumptions of Lemma 4 hold. Then, there exists $c > 0$ and a ℓ^q -bounded $(PS)_c$ sequence for J .*

Proof. By Lemma 4 and Theorem 3 there exists a sequence $(u_m) \subset \ell^q$ such that

$$J(u_m) \rightarrow c \text{ and } J'(u_m) \rightarrow 0,$$

where

$$\begin{aligned} c &= \inf_{\gamma \in \Gamma} \left\{ \max_{t \in [0,1]} J(\gamma(t)) \right\}, \\ \Gamma &= \{ \gamma \in C([0,1], \ell^q) : \gamma(0) = 0, \gamma(1) = e \}, \end{aligned}$$

and e is defined in the proof of Lemma 4.

We will prove that the sequence (u_m) is bounded in ℓ^q . We have for $\mu > p \geq q$

$$\begin{aligned} \langle J'(u_m), u_m \rangle &= \sum_{k \in \mathbb{Z}} |\Delta u_m(k-1)|^p \\ &\quad + \sum_{k \in \mathbb{Z}} V(k) |u_m(k)|^q - \lambda \sum_{k \in \mathbb{Z}} f(k, u_m(k)) u_m(k), \end{aligned}$$

and, by (F_2) ,

$$\begin{aligned} &\mu J(u_m) - \langle J'(u_m), u_m \rangle \\ &= \left(\frac{\mu}{p} - 1 \right) \sum_{k \in \mathbb{Z}} |\Delta u_m(k-1)|^p + \left(\frac{\mu}{q} - 1 \right) \sum_{k \in \mathbb{Z}} V(k) |u_m(k)|^q \\ &\quad - \lambda \sum_{k \in \mathbb{Z}} (\mu F(k, u_m(k)) - f(k, u_m(k)) u_m(k)) \\ &\geq \left(\frac{\mu}{q} - 1 \right) q \|u_m\|_q^q = (\mu - q) \|u_m\|_q^q, \end{aligned}$$

which implies that the sequence u_m is bounded in ℓ^q . \square

Now we are in a position to prove Theorem 1.

Proof of Theorem 1.

For any $m \in \mathbb{N}$, the sequence $\{|u_m(k)|, k \in \mathbb{Z}\}$, given in Lemma 5, is bounded in ℓ^q and, in consequence, $|u_m(k)| \rightarrow 0$ as $|k| \rightarrow \infty$. Let $|u_m(k)|$ takes its maximum at $k_m \in \mathbb{Z}$. There exists a unique $j_m \in \mathbb{Z}$, such that $j_m T \leq k_m < (j_m + 1)T$ and

let $w_m(k) = u_m(k + j_m T)$. Then $|w_m(k)|$ takes its maximum at $i_m = k_m - j_m T \in [0, T - 1]$. By the T -periodicity of V and $f(\cdot, t)$, it follows that

$$\|u_m\|_q = \|w_m\|_q$$

and

$$J(u_m) = J(w_m).$$

Since (u_m) is bounded in ℓ^q , there exists $w \in \ell^q$, such that $w_m \rightharpoonup w$ weakly in ℓ^q . The weak convergence in ℓ^q implies that $w_m(k) \rightarrow w(k)$ for every $k \in \mathbb{Z}$. Indeed, if we take a test function $v_k \in \ell^q$, $v_k(k) = 1$, $v_k(j) = 0$ if $j \neq k$, then

$$w_m(k) = \langle w_m, v_k \rangle \rightarrow \langle w, v_k \rangle = w(k).$$

Moreover, for any $v \in \ell^q$

$$\begin{aligned} |\langle J'(w_m), v \rangle| &= |\langle J'(u_m), v(\cdot + j_m T) \rangle| \\ &\leq \|J'(u_m)\|_* \|v(\cdot + j_m T)\|_q \\ &= \|J'(u_m)\|_* \|v\|_q \rightarrow 0, \end{aligned}$$

which implies that $J'(w_m) \rightarrow 0$, which means that for every $v \in \ell^q$

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \varphi_p(\Delta w_m(k-1)) \Delta v(k-1) + \sum_{k \in \mathbb{Z}} V(k) \varphi_q(w_m(k)) v(k) \\ - \lambda \sum_{k \in \mathbb{Z}} f(k, w_m(k)) v(k) \rightarrow 0, \quad \text{for all } k \in \mathbb{Z}, \quad \text{as } m \rightarrow \infty. \end{aligned} \quad (12)$$

Let us take $v \in \ell^q$ with compact support, i.e., there exist $a, b \in \mathbb{Z}$, $a < b$ such that $v(k) = 0$ if $k \in \mathbb{Z} \setminus [a, b]$ and $v(k) \neq 0$ if $k \in \{a+1, b-1\}$. The set of such elements ℓ_0^q is dense in ℓ^q , because if $v \in \ell^q$ and $v_k \in \ell_0^q$ is such that $v_k(j) = 0$ if $|j| \geq k+1$, $v_k(j) = v(j)$ if $|j| \leq k$, then $\|v - v_k\|_q \rightarrow 0$ as $k \rightarrow \infty$. Taking $v \in \ell_0^q$ in (12), due to the finite sums and the continuity of functions $f(k, \cdot)$, we obtain, passing to a limit, that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \varphi_p(\Delta w(k-1)) \Delta v(k-1) + \sum_{k \in \mathbb{Z}} V(k) \varphi_q(w(k)) v(k) \\ - \lambda \sum_{k \in \mathbb{Z}} f(k, w(k)) v(k) = 0, \quad \text{for all } v \in \ell_0^q. \end{aligned} \quad (13)$$

From the density of ℓ_0^q in ℓ^q , we deduce that the previous equality is fulfilled for all $v \in \ell^q$ and, in consequence, w is a critical point of the functional J , i.e., w is a solution of Eq. (5).

It remains to show that $w \neq 0$.

Assuming, on the contrary, that $w = 0$, we conclude that

$$|u_m|_\infty = |w_m|_\infty = \max\{|w_m(k)| : k \in \mathbb{Z}\} \rightarrow 0, \quad \text{as } m \rightarrow \infty. \quad (14)$$

By (F_3) , for a given $\varepsilon > 0$, there exists $\delta > 0$, such that if $|x| < \delta$ then, for every $k \in [0, T - 1]$, the following inequalities holds

$$\begin{aligned} |F(k, x)| &\leq \varepsilon |x|^q, \\ |f(k, x) x| &\leq \varepsilon |x|^q. \end{aligned} \quad (15)$$

By (14), for every $k \in [0, T - 1]$, there exists a positive integer M_k such that for all $m > M_k$ it follows that $|w_m(k)| < \delta$. Since the maximum value of $|w_m|$ is attained at $i_m \in [0, T - 1]$, it follows that for $m > M = \max\{M_k : k \in [0, T - 1]\}$ and every $k \in \mathbb{Z}$

$$|w_m(k)| \leq |w_m(i_m)| \leq \delta.$$

Then, by (15), for $m > M$ and every $k \in \mathbb{Z}$:

$$\begin{aligned} |F(k, w_m(k))| &\leq \varepsilon |w_m(k)|^q, \\ |f(k, w_m(k)) w_m(k)| &\leq \varepsilon |w_m(k)|^q, \end{aligned} \quad (16)$$

which implies that

$$\begin{aligned} 0 &\leq qJ(w_m) = \frac{q}{p} \sum_{k \in \mathbb{Z}} |\Delta w_m(k-1)|^p + \sum_{k \in \mathbb{Z}} V(k) |w_m(k)|^q \\ &\quad - \lambda \sum_{k \in \mathbb{Z}} qF(k, w_m(k)) \\ &\leq \sum_{k \in \mathbb{Z}} |\Delta w_m(k-1)|^p + \sum_{k \in \mathbb{Z}} V(k) |w_m(k)|^q - \lambda \sum_{k \in \mathbb{Z}} f(k, w_m(k)) w_m(k) \\ &\quad - \lambda \sum_{k \in \mathbb{Z}} (qF(k, w_m(k)) - f(k, w_m(k)) w_m(k)) \\ &\leq \langle J'(w_m), w_m \rangle + \lambda(q\varepsilon |w_m|_q^q + \varepsilon |w_m|_q^q) \\ &\leq \|J'(w_m)\|_* \|w_m\|_q + \lambda\varepsilon \frac{q+1}{V_0} \|w_m\|_q^q. \end{aligned} \quad (17)$$

Since $\|w_m\|$ is bounded in ℓ^q , $J'(w_m) \rightarrow 0$ and ε is arbitrary, by (17) we obtain a contradiction with $J(w_m) = J(u_m) \rightarrow c > 0$. The proof of the first part is complete.

Now, let u be a nonzero homoclinic solution of problem (5). Assume that it attains positive local maximums and/or negative local minimums at infinitely many points k_n . In particular we can assume that $\{|k_n|\} \rightarrow \infty$. In consequence $\Delta_p^2 u(k_n - 1) u(k_n) \leq 0$ and $u(k_n) \rightarrow 0$.

From this, multiplying in (5) by $u(k_n)/|u(k_n)|^q$, we have

$$\lambda \frac{f(k_n, u(k_n)) u(k_n)}{|u(k_n)|^q} \geq \frac{\Delta_p^2 u(k_n - 1) u(k_n)}{|u(k_n)|^q} + \lambda \frac{f(k_n, u(k_n)) u(k_n)}{|u(k_n)|^q} = V(k_n) \geq V_0 > 0. \quad (18)$$

By means of condition (F_3) we arrive at the following contradiction:

$$0 = \lambda \lim_{n \rightarrow \infty} \frac{f(k_n, u(k_n)) u(k_n)}{|u(k_n)|^q} \geq V_0 > 0.$$

Suppose now that function u vanish at infinitely many points l_n . From condition (F_3) we conclude that $\Delta_p^2 u(l_n - 1) = 0$ and, in consequence, $u(l_n - 1) u(l_n + 1) < 0$. Therefore it has an unbounded sequence of positive local maximums and negative local minimums, in contradiction with the previous assertion.

As a direct consequence of the two previous properties, we deduce that, for $|k|$ large enough, function u has constant sign and it is strictly monotone. \square

To illustrate the optimality of the obtained results, we present in the sequel an example in which it is pointed out that condition (F_2) cannot be removed to deduce the existence result proved in Theorem 1.

Example 6 Let $W(k) > 0$ be a T -periodic sequence, $W_1 = \max \{W(k) : k \in [0, T-1]\}$, $p \geq q > 1$ and $r > q$ be fixed. Consider problem (5) with

$$f(k, t) = \begin{cases} W(k) \varphi_r(t), & \text{if } |t| \leq 1 \\ W(k) \varphi_q(t), & \text{if } |t| \geq 1. \end{cases}$$

It is obvious that condition (F_1) holds. Since $r > q$ we have that condition (F_3) is trivially fulfilled. Concerning to condition (F_2) , we have that

$$F(k, t) = \begin{cases} W(k) \frac{|t|^r}{r}, & \text{if } |t| \leq 1 \\ W(k) \left(\frac{|t|^q}{q} + \frac{q-r}{qr} \right), & \text{if } |t| \geq 1. \end{cases}$$

It is clear that $F(k, t) > 0$ for all $t \neq 0$ and that $\mu F(k, t) \leq t f(k, t)$ for all $t \in [-1, 1]$ if and only if $0 < \mu \leq r$.

When $|t| \geq 1$, the inequality $\mu F(k, t) \leq t f(k, t)$ holds if and only if either $\mu = q$ or $\mu > q$ and

$$|t|^q \leq \frac{\mu(r-q)}{r(\mu-q)} < \infty.$$

As consequence, the inequality $\mu F(k, t) \leq t f(k, t)$ for all $t \neq 0$ is satisfied if and only if $\mu = q$, i. e. condition (F_2) does not hold.

Let's see that this problem has only the trivial solution for small values of the parameter λ .

Since $r > q$, it is not difficult to verify that, for $0 < \lambda < \frac{(q-1)V_0}{(r-1)W_1}$, the function $\lambda f(k, t) - V(k) \varphi_q(t)$ is strictly decreasing for every integer k . So, for λ in that situation, we have that

$$(\lambda f(k, t) - V(k) \varphi_q(t)) t < 0 \quad \text{for all } t \neq 0 \quad \text{and all } k \in \mathbb{Z}.$$

Suppose that there is a nontrivial solution u of the considered problem, and moreover it takes some positive values. Let k_0 be such that $u(k_0) = \max \{u(k); k \in \mathbb{Z}\} > 0$. In such a case we deduce the following contradiction:

$$0 = \Delta_p^2 u(k_0 - 1) - V(k_0) \varphi_q(u(k_0)) + \lambda f(k_0, u(k_0)) < \Delta_p^2 u(k_0 - 1) \leq 0.$$

Analogously it can be verified that the solution u has no negative values on \mathbb{Z} . \square

3 Remarks and examples

In this section we will consider some examples and remarks on applications and extensions of Theorem 1 to the existence of homoclinic solutions of difference equations of following types:

(A) Second-order discrete p -Laplacian equations of the form

$$\Delta_p^2 u(k-1) - V(k) u(k) |u(k)|^{q-2} + \lambda b(k) u(k) |u(k)|^{r-2} = 0, \quad (19)$$

with $r > p \geq q > 1$.

(B) Higher even-order difference equations. A model equation is the fourth-order extended Fisher-Kolmogorov equation

$$\Delta^4 u(k-2) - a\Delta^2 u(k-1) + V(k)u(k)|u(k)|^{q-2} - \lambda b(k)u(k)|u(k)|^{r-2} = 0, \quad (20)$$

with $r > q > 1$.

(C) Second-order difference equations with cubic and quintic nonlinearities of the forms

$$\Delta^2 u(k-1) - V(k)u(k) + \lambda(b(k)u^3(k) + c(k)u^5(k)) = 0, \quad (21)$$

and

$$\Delta_p^2 u(k-1) - a(k)u(k) + \lambda(b(k)u^2(k) + c(k)u^3(k)) = 0, \quad (22)$$

arising in mathematical physics and biology.

(A) *Second-order discrete p -Laplacian equations.*

The spectrum of the Dirichlet problem (D_N) for Eq. (19), subject to Dirichlet boundary conditions

$$u(0) = u(N+1) = 0,$$

is studied in [11]. It is proved that if $2 < r < q$, $N \geq 2$ and $b : [1, N] \rightarrow (0, \infty)$ is a given function, then there exist two positive constants $\lambda_0(N)$ and $\lambda_1(N)$ with $\lambda_0(N) \leq \lambda_1(N)$ such that no $\lambda \in (0, \lambda_0(N))$ is an eigenvalue of problem (D_N) while any $\lambda \in [\lambda_1(N), \infty)$ is an eigenvalue of problem (D_N) . Moreover, we have

$$\lambda_1(N) \leq \frac{r}{2}\lambda_0(N) \quad \text{and} \quad \frac{4}{(N+1)^2|b|_\infty} \leq \lambda_0(N) \leq \lambda_1(N) \leq B(r, q, b, N),$$

where $B(r, q, b, N) = \frac{r(q-2)}{(q-r)\sum_{k=1}^N b(k)} \left(\frac{N(q-r)}{q(r-2)} \right)^{\frac{r-2}{q-2}}$ and $|b|_\infty = \max_{k \in [1, N]} b(k)$. Note that if $b(k)$ is positive and $b(k) \geq b > 0$ then

$$B(r, q, b, N) \leq KN^{\frac{r-q}{q-2}},$$

where K is a constant depending on p, q, b , which implies that $\lambda_0(N) \rightarrow 0$ and $\lambda_1(N) \rightarrow 0$ as $N \rightarrow \infty$. It implies that for a given $\varepsilon > 0$, there exists N_0 such that for any $N > N_0$, the problem (D_N) has a solution for every $\lambda > \varepsilon > 0$.

We extend this phenomenon, looking for homoclinic solutions of Eq. (19). Applying Theorem 1 with $f(k, t) = b(k)\varphi_r(t)$, $F(k, t) = b(k)\Phi_r(t)$ and $\mu = r > p \geq q > 1$, we obtain

Corollary 7 *Suppose that the function $V : \mathbb{Z} \rightarrow \mathbb{R}$ is positive and T -periodic and $r > p \geq q > 1$. Then, for each $\lambda > 0$, Eq. (19) has a nonzero homoclinic solution.*

Moreover, given a nontrivial solution u of problem (19), there exist k_\pm two integer numbers such that for all $k > k_+$ and $k < k_-$, the sequence $u(k)$ is strictly monotone.

(B) *Higher even-order difference equations.*

The statement of Theorem 1 can be extended to higher even-order difference equations. For simplicity we consider the fourth-order difference equations of the form

$$\Delta^2(\varphi_{p_2}(\Delta^2 u(k-2))) - a\Delta(\varphi_{p_1}(\Delta u(k-1))) + V(k)\varphi_q(u(k)) - \lambda f(k, u(k)) = 0, \quad (23)$$

where $f(k, \cdot) \in C(\mathbb{R}, \mathbb{R})$ for each $k \in \mathbb{Z}$, satisfy the assumptions (F1) – (F3).

We consider the functional $J_1 : \ell^q \rightarrow \mathbb{R}$,

$$J_1(u) = A_1(u) - \lambda \sum_{k \in \mathbb{Z}} F(k, u(k)),$$

where

$$A_1(u) = \sum_{k \in \mathbb{Z}} \Phi_{p_2}(\Delta^2 u(k-2)) + a \Phi_{p_1}(\Delta u(k-1)) + V(k) \Phi_q(u(k)),$$

which is well defined for $\mu > p_j \geq q > 1$, $j = 1, 2$.

Note that the series $\sum_{k \in \mathbb{Z}} \Phi_{p_2}(\Delta^2 u(k-2))$ is convergent because

$$\begin{aligned} \Phi_{p_2}(\Delta^2 u(k-2)) &= \Phi_{p_2}(u(k) - 2u(k-1) + u(k-2)) \\ &\leq \frac{2 \cdot 3^{p_2-1}}{p_2} (|u(k)|^{p_2} + |u(k-1)|^{p_2} + |u(k-2)|^{p_2}), \end{aligned}$$

and

$$\sum_{k \in \mathbb{Z}} \Phi_{p_2}(\Delta^2 u(k-2)) \leq \frac{2 \cdot 3^{p_2}}{p_2} \sum_{k \in \mathbb{Z}} |u(k)|^{p_2},$$

while $\sum_{k \in \mathbb{Z}} |u(k)|^{p_2}$ is convergent since $q \leq p_2$.

Now following the steps of the proof of Theorem 1 one can prove

Theorem 8 *Suppose that $a > 0$, the function $V : \mathbb{Z} \rightarrow \mathbb{R}$ is positive and T -periodic and the functions $f(k, \cdot) : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy assumptions (F1)-(F3) and $\mu > p_j \geq q > 1$, $j = 1, 2$. Then, for each $\lambda > 0$, Eq. (23) has a nonzero homoclinic solution $u \in \ell^q$, which is a critical point of the functional $J_1 : \ell^q \rightarrow \mathbb{R}$.*

A typical example of Eq. (23) is Eq. (20), which is a discretization of a fourth-order extended Fisher-Kolmogorov equation. Homoclinic solutions for fourth-order ODEs are studied in [20] using variational approach and concentration-compactness arguments. As a consequence of Theorem 8 we obtain

Corollary 9 *Suppose that $a > 0$, the function $V : \mathbb{Z} \rightarrow \mathbb{R}$ is positive and T -periodic and $r > q > 1$. Then, for each $\lambda > 0$, Eq. (20) has a nonzero homoclinic solution $u \in \ell^q$.*

(C) *Second-order difference equations with cubic and quintic nonlinearities*

Our next example is Eq. (21), known as stationary Ginzburg-Landau equation with cubic-quintic nonlinearity. We refer to [6], [14] and references therein. From physical point of view it is interesting the case $b(k) = b, c(k) = c, b, c < 0$. Theorem 1 can be applied for $f(k, t) = b(k)t^3 + c(k)t^5$ with $b(k), c(k)$, T -periodic and $c(k)$ positive. Then $f(k, t)$ satisfies assumptions (F1) – (F3) with $\mu = 4$ and as a consequence we have

Corollary 10 *Suppose that the functions $V : \mathbb{Z} \rightarrow \mathbb{R}$, $b : \mathbb{Z} \rightarrow \mathbb{R}$ and $c : \mathbb{Z} \rightarrow \mathbb{R}$ are T -periodic and V and c are positive. Then, for each $\lambda > 0$, the equation (21) has a nonzero homoclinic solution $u \in \ell^q$.*

Moreover, given a nontrivial solution u of problem (5), there exist k_{\pm} two integer numbers such that for all $k > k_+$ and $k < k_-$, the sequence $u(k)$ is strictly monotone.

Moreover, we can prove that if in addition to conditions $(F_1) - (F_3)$ the following condition holds:

$$(F_4) \quad f(k, t) > 0 \text{ for all } t < 0 \text{ and all } k \in \mathbb{Z},$$

the homoclinic solution of Eq. (5) is positive.

Indeed, let u be a homoclinic solution of Eq. (5) and assume that (F_4) holds. Suppose that there exists k_0 such that $u(k_0) < 0$ and let k_1 be such that $u(k_1) = \min \{u(k), k \in \mathbb{Z}\} < 0$. In consequence $\Delta_p^2 u(k_1 - 1) \geq 0$, which implies that

$$\lambda f(k_1, u(k_1)) = -\Delta_p^2 u(k_1 - 1) + V(k_1) \varphi_q(u(k_1)) < 0,$$

in contradiction with (F_4) . Then $u(k) \geq 0$ for every $k \in \mathbb{Z}$.

If $u(k_2) = 0$ for some $k_2 \in \mathbb{Z}$, we know that $\Delta_p u(k_2 - 1) = 0$ and, in consequence, $u(k_2 - 1)u(k_2 + 1) < 0$, and we arrive at a contradiction as in the previous case.

We summarize above observations in

Theorem 11 *Suppose that the function $V : \mathbb{Z} \rightarrow \mathbb{R}$ is positive and T -periodic and the functions $f(k, \cdot) : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy assumptions (F_1) , (F_2) and (F_3) . Then, for each $\lambda > 0$, the equation (5) has a nonzero homoclinic solution $u \in \ell^q$. If moreover (F_4) holds, u is a positive solution on \mathbb{Z} that is strictly monotone for $|k|$ large enough.*

In the case $q = 2$ we can estimate the maximum of the solution u , provided the additional assumption

(F_5) Assume that for all $t > 0$ and $k \in \mathbb{Z}$ function $f(k, \cdot)$ has the form $f(k, t) = t g(k, t)$, where $g(k, t)$ is T -periodic in k , $g(k, 0) = 0$ and for each k , $g(k, t)$ is increasing in t for $t > 0$.

Let $g^{-1}(k, t)$ be the inverse function of $g(k, t)$ for $t > 0$. We have that $g^{-1}(k, t)$ is increasing in t for $t > 0$. Let u be a positive homoclinic solution of the equation (5) in view of last theorem and $u(k_0) > 0$ is its maximum. Note that, in view of the periodicity of coefficients, if $u(\cdot)$ is a solution of Eq. (5), then $u(\cdot + jT)$, $j \in \mathbb{Z}$ is also a solution of Eq. (5). Hence, we may assume that $k_0 \in [0, T - 1]$. Then $\Delta_p^2 u(k_0 - 1) \leq 0$ and

$$\lambda u(k_1)g(k_1, u(k_1)) - V(k_1)u(k_1) \geq 0,$$

and hence by properties of g and V

$$u(k_1) \geq g^{-1}\left(k_1, \frac{V_0}{\lambda}\right).$$

Thus

$$\max\{u(k) : k \in [0, T - 1]\} \geq \min\{g^{-1}(k, \frac{V_0}{\lambda}) : k \in [0, T - 1]\}. \quad (24)$$

We summarize above observation in

Corollary 12 *Let $q = 2$ and suppose that the functions $V : \mathbb{Z} \rightarrow \mathbb{R}$ and $f : \mathbb{Z} \rightarrow \mathbb{R}$ satisfy assumptions of Theorem 11. Then, if in addition, f satisfies condition (F_5) , the positive homoclinic solution of the equation*

$$\Delta_p^2 u(k - 1) - V(k)u(k) + \lambda u(k)g(k, u(k)) = 0,$$

satisfies the estimate (24).

Our next example, concerning Theorem 11, are the equation (22) and

$$\Delta_p^2 u(k-1) - a(k)u(k) + \lambda(b(k)u^2(k) + c(k)u_+^3(k)) = 0, \quad (25)$$

where $u_+ = \max\{u, 0\}$.

Positive homoclinic solutions of corresponding differential equation are studied in [9] and periodic solutions in [8]. We suppose that the coefficients $a(k), b(k)$ and $c(k)$ are T -periodic and there are constants a, b, B, c and C such that

$$0 < a \leq a(k), \quad 0 \leq b \leq b(k) \leq B, \quad 0 < c \leq c(k) \leq C, \quad \forall k \in [0, T-1]. \quad (26)$$

By Theorem 11, Eq. (25) has a positive solution u , which is a critical point of the functional $I : l^2 \rightarrow \mathbb{R}$,

$$I(u) = \sum_{k \in \mathbb{Z}} \Phi_p(\Delta u(k-1)) + \frac{1}{2} \sum_{k \in \mathbb{Z}} a(k)u^2(k) - \lambda \sum_{k \in \mathbb{Z}} \left(\frac{1}{3} b(k)u^3(k) + \frac{1}{4} c(k)u_+^4(k) \right).$$

Clearly, the positive solution of Eq. (25) is a positive solution of Eq. (22) too.

Further, let u takes its positive maximum at $k_1 \in [0, T-1]$, then $\Delta_p^2 u(k_1-1) \leq 0$ and, since $u(k_1) > 0$, we have from Eq. (22) that

$$-a(k_1) + \lambda(b(k_1)u(k_1) + c(k_1)u^2(k_1)) \geq 0.$$

In view of (26), the last inequality implies

$$u(k_1) \geq \frac{-\lambda b(k_1) + \sqrt{\lambda^2 b(k_1)^2 + 4\lambda a(k_1)c(k_1)}}{2\lambda c(k_1)},$$

or

$$\max\{u(k) : k \in [0, T-1]\} \geq \frac{-\lambda B + \sqrt{\lambda^2 b^2 + 4\lambda a c}}{2\lambda C} = \frac{-B + \sqrt{b^2 + 4ac/\lambda}}{2C}. \quad (27)$$

We obtain a positive lower bound for $\max\{u(k) : k \in [0, T-1]\}$ in the case

$$0 < \lambda < \frac{4ac}{B^2 - b^2}$$

and (27) shows that $\max\{u(k) : k \in [0, T-1]\}$ blows up, i.e. tends to $+\infty$ as $\lambda \rightarrow 0$.

We summarize above facts in

Corollary 13 *Let $p > 1, \lambda > 0$ and $a(k), b(k)$ and $c(k)$ be T -periodic sequences. Assume that there are constants a, b, B, c and C such that (26) holds. Then, the equation*

$$\Delta_p^2 u(k-1) - a(k)u(k) + \lambda(b(k)u^2(k) + c(k)u^3(k)) = 0,$$

has a positive homoclinic solution and for $0 < \lambda < \frac{4ac}{B^2 - b^2}$,

$$\max\{u(k) : k \in [0, T-1]\} \geq \frac{-B + \sqrt{b^2 + 4ac/\lambda}}{2C} > 0.$$

Let $\lambda_m \rightarrow 0$. By the last statement, if u_m is the solution of the equation

$$\Delta_p^2 u(k-1) - a(k)u(k) + \lambda_m(b(k)u^2(k) + c(k)u^3(k)) = 0,$$

then

$$\lim_{m \rightarrow \infty} \max\{u_m(k) : k \in [0, T-1]\} = +\infty.$$

Let $k_m \in [0, T-1]$ be such that $u_m(k_m) = \max\{u_m(k) : k \in [0, T-1]\}$. Since k_m is an infinite sequence of integers, by Dirichlet principle, there exists a fixed $k_* \in [0, T-1]$ and a subsequence of u_m , still denoted by u_m , such that $u_m(k_m) = u_m(k_*)$ and $\lim_{m \rightarrow \infty} u_m(k_*) = +\infty$. Note that if $T = 2$, then $k_* = 0$ or $k_* = 1$.

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References

- [1] R. P. Agarwal, K. Perera and D. O'Regan, Multiple positive solutions of singular and nonsingular discrete problems via variational methods, *J. Nonlinear Analysis*, **58** (2004), 69-73.
- [2] H. Brezis, L. Nirenberg, Remarks on finding critical points, *Comm. Pure Appl. Math.*, **44** (1991) 939-963.
- [3] A. Cabada, A. Iannizzotto, S. Tersian, Multiple solutions for discrete boundary value problems, *J. Math. Anal. Appl.*, **356** (2009) 418-428.
- [4] P. Chen, H. Fang, Existence of Periodic and Subharmonic Solutions for Second-Order p-Laplacian Difference Equations, *Advances in Difference Equations*, 2007, Article ID 42530, doi:10.1155/2007/42530
- [5] V. Coti-Zelati, P. H. Rabinowitz, Homoclinic orbits for Hamiltonian systems possessing superquadratic potentials, *J. Amer. Math. Soc.*, **4** (1991) 693-727.
- [6] N. K. Efremidis, D. N. Christodoulides, Discrete Ginzburg-Landau solitons, *Physical Review E*, **67**, 026606 (2003).
- [7] H. Fang, D. Zhao, Existence of nontrivial homoclinic orbits for fourth-order difference equations, *Applied Mathematics and Computation*, **214** (2009) 163-170.
- [8] M. R. Grossinho, L. Sanchez, A note on periodic solutions of some nonautonomous differential equations, *Bull. Austr. Math. Soc.*, **34** (1986) 253-265.
- [9] M. R. Grossinho, F. Minhos, S. Tersian, Positive homoclinic solutions for a class of second order differential equations, *J. Math. Anal. Appl.*, **240(1)** (1999) 163-173.
- [10] L. Jiang, Z. Zhou, Three Solutions to Dirichlet Boundary Value Problems for p-Laplacian Difference Equations, *Advances in Difference Equations*, 2008, Article ID 345916, doi:10.1155/2008/345916.
- [11] A. Kristály, M. Mihăilescu, V. Rădulescu, S. Tersian, Spectral estimates for a nonhomogeneous difference problem, CEU math preprints, November 2008, <http://web.ceu.hu/math/Research/Preprints>.
- [12] C. Li, A unified proof on existence of homoclinic orbits for some semilinear ordinary differential equations with periodic potentials, *J. Math. Anal. Appl.*, **365** (2010) 510-516.
- [13] M. Ma, Z. Guo, Homoclinic orbits for second order self-adjoint difference equations, *J. Math. Anal. Appl.*, **323** (2006) 513-521.
- [14] K.-I. Maruno, A. Ankiewicz, N. Akhmediev, Dissipative solitons of the discrete complex cubic-quintic Ginzburg-Landau equation, *Physics Letters A*, **347** (2005) 231-240.

- [15] M. Mihăilescu, V. Rădulescu and S. Tersian, Eigenvalue problems for anisotropic discrete boundary value problems, *Journal of Difference Equations and Applications*, **15**, No. 6, June 2009, 557–567 .
- [16] W. Omana, M. Willem, Homoclinic orbits for a class of Hamiltonian systems, *Differential Integral Equations*, **5** (5) (1992) 1115–1120.
- [17] H. Poincaré. Les méthodes nouvelles de la mécanique céleste, Gauthier-Villars, Paris, 1899.
- [18] P. H. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, *Proc. Roy. Soc. Edinburgh Sect. A*, **114** (1990) 33–38.
- [19] K. Tanaka, Homoclinic orbits in a first-order superquadratic Hamiltonian system: Convergence of subharmonic orbits, *J. Differential Equations*, **94**(1991) 315-339.
- [20] S. Tersian, J. Chaparova, Periodic and homoclinic solutions of extended Fisher–Kolmogorov equations, *J. Math. Anal. Appl.*, **260** (2001) 490–506.

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