MULTI PARAMETER PROXIMAL POINT ALGORITHMS

O. A. BOIKANYO AND G. MOROŞANU

Abstract. The aim of this paper is to prove a strong convergence result associated with an algorithm introduced by Y. Yao and M. A. Noor (2008) under a new control condition on one of the parameters involved. Further, convergence properties of a generalized proximal point algorithm which was introduced in [4] are analyzed. The results of this paper are proved under the general condition that errors tend to zero in norm. These results extend and improve some recently announced results concerning the regularization method and the proximal point algorithm.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced Hilbertian norm $\| \cdot \|$. Recall that a map $T : H \to H$ is said to be nonexpansive if for every $x, y \in H$ the inequality $\|Tx - Ty\| \leq \|x - y\|$ holds. In the case when $\|Tx - Ty\| \leq a\|x - y\|$ holds for some $a \in (0, 1)$, then T is said to be a contraction with Lipschitz constant a. The map T is called firmly nonexpansive if for any $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle.$$

Obviously, firmly nonexpansive mappings are nonexpansive. An operator $A : D(A) \subset H \to 2^H$ is said to be monotone if

$$\langle x - x', y - y' \rangle \geq 0, \quad \forall\, (x, y), (x', y') \in G(A).$$

That is, its graph $G(A) = \{(x, y) \in H \times H : x \in D(A), y \in Ax\}$ is a monotone subset of $H \times H$. An operator A is called maximal monotone if in addition to being monotone, its graph is not properly contained in the graph of any other monotone operator. In nonlinear analysis and convex optimization, an important and perhaps interesting topic is to find zeros of maximal monotone operators. From the view point of fixed point theory, the problem

$$\text{find an } x \in D(A) \text{ such that } 0 \in A(x), \tag{1}$$

is equivalent to finding fixed points of the firmly nonexpansive mapping $(I + \lambda A)^{-1} : H \to H$, the so called resolvent of A for all $\lambda > 0$. Indeed, many problems that involve convexity can be formulated as finding zeroes of maximal monotone operators. Such problems include, but are not limited to convex minimization, variational inequalities and concave-convex mini-max problems.

One of the most popular and powerful solution techniques for solving nonlinear problems is the so called proximal point algorithm which was initially suggested by Martinet [9] and later extensively developed by Rockafellar [10]. The main idea of this method is to replace the original problem (1) by a sequence of regularized problems

$$\text{find an } x \in D(A) \text{ such that } 0 \in A(x) + \beta_n^{-1}(x - x_n), \tag{2}$$

so that at each step, problem (2) has a unique solution $x := x_{n+1}$. Here (β_n) is a sequence of positive real numbers and $x_0 \in H$ is a given starting point. Accordingly, x_{n+1} solves problem (2) if and only if

$$x_{n+1} = J_{\beta_n} x_n, \quad \text{where } J_{\beta_n} = (I + \beta_n A)^{-1}.$$

2000 *Mathematics Subject Classification.* 47J25, 47H05, 47H09.

Key words and phrases. maximal monotone operator, proximal point algorithm, nonexpansive map, firmly nonexpansive operator.

In general, equation (2) is solved only approximately, in which case, x_{n+1} is the inexact solution of problem (2), i.e.,

$$x_{n+1} = J_{\beta_n} x_n + e_n, \quad n = 0, 1 \ldots, \tag{3}$$

where (e_n) is considered to be the sequence of computational errors. Rockafellar [10] proved the weak convergence of algorithm 3 to an element of the fixed point set $F(J_c) = \{x \in H \,|\, J_c x = x\} = A^{-1}(0)$, for all $c > 0$, provided that this set is not empty, with the conditions $\liminf_{n\to\infty} \beta_n > 0$ and

$$(E1) \ \sum_{n=0}^{\infty} \|e_n\| < \infty$$

being satisfied. He also proved strong convergence if in addition, the operator $A^{-1}$is Lipschitz continuous at zero (with modulus $a \geq 0$), that is, $A^{-1}(0) = \{y\}$, and for some $\tau > 0$, $\|z - y\| \leq a \|z'\|$ whenever $(z, z') \in G(A)$ and $\|z'\| \leq \tau$. Güler's example [6] which revealed that the PPA fails in general to converge strongly, gave rise to the natural question(s): how can the PPA be modified so as to obtain strong convergence? Or is it possible to construct/design strongly convergent proximal point algorithms? There are several attempts made so far in order to address the above question(s). One such attempt was made in [11] by Solodov and Svaiter. Another effort was made independently by Xu [14], and Kamimura and Takahashi [7]. They proposed the following inexact PPA which is simpler than the one obtained by Solodov and Svaiter

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{\beta_n} x_n + e_n, \quad \text{for any } u, x_0 \in H \text{ and all } n \geq 0, \tag{4}$$

where $(\alpha_n) \subset (0, 1)$ and $(\beta_n) \subset (0, \infty)$. In fact, their version of the PPA corresponds to the case when $u = x_0$, the initial starting point of the trajectory generated by (4). They proved a strong convergence result when (e_n) satisfies (E1). The case when (e_n) is not summable and u not necessarily the starting point of the PPA was treated in [1]. A strong convergence result was obtained in [2] by further weakening the error condition of [1]. It should however be mentioned that unlike in [1], such a condition depend on the sequence of parameters (α_n). The precise condition on the sequence of error terms (e_n) was

$$(E2) \ \lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0.$$

Another algorithm of interest which generates strongly convergent sequences is the regularization method which was introduced by Lehdili and Moudafi [8], and extended by Xu [15]. Given $x_0, u \in H$, this method according to Xu generates a sequence (x_n) iteratively by

$$x_{n+1} = J_{\beta_n}(\alpha_n u + (1 - \alpha_n) x_n + e_n), \quad \text{for all } n \geq 0. \tag{5}$$

The authors have observed in [2] (see also [3]) that there is a strong connection between the proximal point algorithm (4) and the regularization method (5). More precisely, they noted that taking

$$v_n = \frac{x_n - \alpha_{n-1} u - e_{n-1}}{1 - \alpha_{n-1}},$$

equation (4) reduces to

$$v_{n+1} = J_{\beta_n}(\alpha_{n-1} u + (1 - \alpha_{n-1}) v_n + e_{n-1}), \quad \text{for all} \quad n \geq 1, \tag{6}$$

and for $\alpha_n \to 0$ and $e_n \to 0$, (x_n) defined by (4) converges if and only if (v_n) does. Thus (4) and (6) are equivalent. The regularization method was further extended [4] to

$$v_{n+1} \ = \ J_{\beta_n}(\alpha_{n-1} u + \lambda_{n-1} v_n + \gamma_{n-1} T v_n + e_{n-1}), \quad n \geq 1, \tag{7}$$

where $T : H \to H$ is a nonexpansive map, $\alpha_n \in (0, 1)$, $\lambda_n, \gamma_n \in [0, 1]$ with $\alpha_n + \lambda_n + \gamma_n = 1$, and $\beta_n \in (0, \infty)$. Under appropriate conditions on the control parameters $\alpha_n, \lambda_n, \gamma_n$ and β_n, it was shown [4] that (v_n) generated by (7) converges strongly to $P_{A^{-1}(0)} u$, provided that $\emptyset \neq A^{-1}(0) \subset F(T)$, where $F(T) = \{x \in H \mid Tx = x\}$ is the fixed point set of T.

Recently, Yao and Noor [16] proposed an algorithm whose $(n+1)$th iterate, x_{n+1}, is defined by

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \gamma_n J_{\beta_n} x_n + e_n, \quad n \geq 0, \tag{8}$$

where again $u, x_0 \in H$ are given, $\alpha_n \in (0,1)$, $\lambda_n, \gamma_n \in [0,1]$ with $\alpha_n + \lambda_n + \gamma_n = 1$, and $\beta_n \in (0,\infty)$. They showed that (x_n) is strongly convergent to $P_{A^{-1}(0)}u$, provided that $\emptyset \neq A^{-1}(0)$, and the conditions $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$, $\liminf_{n\to\infty} \beta_n > 0$ with

$$(C1)\ \lim_{n\to\infty}(\beta_{n+1} - \beta_n) = 0,$$

and $\lim_{n\to\infty} \alpha_n = 0$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$ are satisfied. We note that for unbounded (β_n), condition (C1) fails to satisfy the natural choice $\beta_n = n$ for all $n \in \mathbb{N}$. Yao and Noor's result brings us to the following question: Does Theorem 4 [16] remains true if (λ_n) is only bounded from above away from 1 and/or (β_n) satisfies weaker conditions which include choices such as $\beta_n = n$ for all $n \in \mathbb{N}$?

The purpose of this paper is in two folds - to address the above question and to discuss the strong convergence of sequences generated by algorithm (7). Our main result is Theorem 12 which is concerned with the following conditions: $\liminf_{n\to\infty} \beta_n > 0$ and

$$(C2)\ \lim_{n\to\infty} \frac{\beta_{n+1}}{\beta_n} = 1.$$

2. Preliminaries

Let $T : H \to H$ be a nonexpansive map, and let $A : D(A) \subset H \to 2^H$ be a maximal monotone operator. Fix $n \in \mathbb{N}$, and define a map $f_n : H \to H$ by the rule $x \mapsto J_{\beta_n}(\alpha_n u + \lambda_n x + \gamma_n Tx + e_n)$, where $\beta_n > 0$, (α_n), (λ_n) and (γ_n) are real sequences in $(0,1)$ such that $\alpha_n + \lambda_n + \gamma_n = 1$, and $u, e_n \in H$ are given. Then one can easily check that f_n is a contraction. Therefore it follows from the Banach contraction principle that f_n has a unique fixed point z_n, say. In other words,

$$z_n = J_{\beta_n}(\alpha_n u + \lambda_n z_n + \gamma_n T z_n + e_n), \quad n \geq 0. \tag{9}$$

We prove the convergence result associated with the sequence (z_n).

Lemma 1. *Let $\beta_n \in (0,\infty)$, and $\alpha_n, \lambda_n, \gamma_n \in (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$, and $\alpha_n + \lambda_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Assume that $\emptyset \neq A^{-1}(0) =: F \subset F(T)$, where $T : H \to H$ is a nonexpansive map, and either $\sum_{n=0}^{\infty} \|e_n\| < \infty$ or $\|e_n\|/\alpha_n \to 0$. Then for any fixed $u, x_0 \in H$, the sequence (z_n) generated by (9) converges strongly to $P_F u$, the projection of u on F.*

Proof. To show that (z_n) is bounded, we first note that if $(\|e_n\|/\alpha_n)$ is bounded, then there exists a positive constant C such that

$$\sup_{n\in\mathbb{N}} \left(\|u - p\| + \frac{\|e_n\|}{\alpha_n} \right) \leq C.$$

For every $p \in F$, we have from (9)

$$\begin{aligned} \|z_n - p\| &\leq \|\alpha_n(u - p + e_n/\alpha_n) + \lambda_n(z_n - p) + \gamma_n(Tz_n - p)\| \\ &\leq \alpha_n \left(\|u - p\| + \frac{\|e_n\|}{\alpha_n} \right) + \lambda_n \|z_n - p\| + \gamma_n \|Tz_n - p\| \\ &\leq (1 - \alpha_n)\|z_n - p\| + \alpha_n C, \end{aligned}$$

where the first two inequalities follow from the fact that J_{β_n} and T are nonexpansive. The last estimate clearly shows that (z_n) is bounded.

Let $\omega_{\mathtt{w}}((z_n))$ be the weak $\omega-$limit set of (z_n). That is,

$$\omega_{\mathtt{w}}((z_n)) = \{y \in H \mid z_{n_k} \rightharpoonup y \text{ for some subsequence } (z_{n_k}) \text{ of } (z_n)\}.$$

We claim that $\omega_{\mathtt{w}}((z_n)) \subset F$. Let (z_{n_j}) be a subsequence of (z_n) converging weakly to some z_∞. Since (λ_{n_j}) is bounded, it has a convergent subsequence, again denoted (λ_{n_j}). There are two possibilities here: either $\lambda_{n_j} \to 1$, or $\lambda_{n_j} \to \theta \in [0,1)$. In the first case, we derive from

$$Az_{n_j} \ni \frac{\alpha_{n_j} u + (\lambda_{n_j} - 1) z_{n_j} + \gamma_{n_j} T z_{n_j} + e_{n_j}}{\beta_{n_j}} \to 0, \quad \text{as} \quad j \to \infty, \tag{10}$$

that $z_\infty \in F$. In the second case, we note that from (9), we have

$$(1-\lambda_n)\langle z_n - Tz_n, z_n - p\rangle + \beta_n \langle Az_n, z_n - p\rangle = \alpha_n \langle u - Tz_n + e_n/\alpha_n, z_n - p\rangle,$$

where $p \in F$. Using the monotonicity of A, we have for some $M > 0$

$$\begin{aligned} \alpha_n M &\geq 2(1-\lambda_n)\langle z_n - Tz_n, z_n - p\rangle \\ &= (1-\lambda_n)(\|z_n - Tz_n\|^2 + \|z_n - p\|^2 - \|Tz_n - Tp\|^2) \\ &\geq (1-\lambda_n)\|z_n - Tz_n\|^2 . \end{aligned}$$

Passing to the limit in the above estimate, with $n = n_j$, we get

$$\lim_{j\to\infty} \|z_{n_j} - Tz_{n_j}\| = 0.$$

Again from (10), we derive $z_\infty \in F$, showing that $\omega_{\mathtt{w}}((z_n)) \subset F$. Therefore, there exists a subsequence (z_{n_k}) of (z_n) converging weakly to $z \in F$ such that

$$\limsup_{n\to\infty} \langle u - P_F u, z_n - P_F u\rangle = \lim_{k\to\infty} \langle u - P_F u, z_{n_k} - P_F u\rangle = \langle u - P_F u, z - P_F u\rangle \leq 0.$$

On the other hand,

$$\begin{aligned} \|z_n - P_F u\|^2 &\leq \alpha_n^2 \left(\|u - P_F u\| + \frac{\|e_n\|}{\alpha_n} \right)^2 + (\lambda_n \|z_n - P_F u\| + \gamma_n \|Tz_n - P_F u\|)^2 \\ &+ 2\alpha_n \left\langle u - P_F u + \frac{e_n}{\alpha_n}, \lambda_n (z_n - P_F u) + \gamma_n (Tz_n - P_F u) \right\rangle \\ &\leq (1-\alpha_n)^2 \|z_n - P_F u\|^2 + \alpha_n^2 \left(\|u - P_F u\| + \frac{\|e_n\|}{\alpha_n} \right)^2 \\ &+ 2\alpha_n \left\langle u - P_F u + \frac{e_n}{\alpha_n}, (1-\alpha_n)(z_n - P_F u) + \gamma_n (Tz_n - z_n) \right\rangle, \end{aligned}$$

where the second inequality follows from the nonexpansivity of T. Hence for some positive constant C^*,

$$(2-\alpha_n)\|z_n - P_F u\|^2 \leq \alpha_n C^* + 2\left\langle u - P_F u + \frac{e_n}{\alpha_n}, (z_n - P_F u) + \gamma_n (Tz_n - z_n) \right\rangle.$$

Passing to the limit in the above inequality, we deduce strong convergence of (z_n) to $P_F u$ as claimed. We leave it to the reader to verify the result in the case when $(\|e_n\|) \in \ell^1$. ■

We note that Lemma 1 above contains Theorem 1 [3] as a special case.

We next recall some lemmas which will be used in the sequel.

Lemma 2 (Suzuki [12]). *Let (x_n) and (y_n) be bounded sequences in a real Banach space and let (ρ_n) be a sequence in $(0,1)$, with $0 < \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n < 1$. Suppose $x_{n+1} = \rho_n y_n + (1-\rho_n) x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 3 (Gobel and Kirk [5]). *A map $T : H \to H$ is firmly nonexpansive if and only if $2T - I$ (where I is the identity map) is nonexpansive.*

Lemma 4 (H. K. Xu [14]). *Let (s_n) be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-a_n)s_n + a_n b_n + c_n, \quad n \geq 0,$$

where (a_n), (b_n), (c_n) satisfy the conditions: (i) $(a_n) \subset [0,1]$, *with* $\sum_{n=0}^{\infty} a_n = \infty$, (ii) $c_n \geq 0$ *for all $n \geq 0$ with* $\sum_{n=0}^{\infty} c_n < \infty$, *and* (iii) $\limsup_{n\to\infty} b_n \leq 0$. *Then* $\lim_{n\to\infty} s_n = 0$.

Concerning the boundedness of the sequences (x_n) and (v_n) defined by (8) and (7), we have the following two lemmas, respectively. The proof Lemma 6 is contained in the proof of Theorem 5 [4].

Lemma 5. [4] *Let $\beta_n \in (0,\infty)$, $\alpha_n \in (0,1)$, and $\lambda_n, \gamma_n \in [0,1]$ with $\alpha_n + \lambda_n + \gamma_n = 1$ for all n. Assume that $F := A^{-1}(0) \neq \emptyset$, and either $\sum_{n=0}^{\infty} \|e_n\| < \infty$ or $(\|e_n\|/\alpha_n)$ is bounded. Then for any fixed $u, x_0 \in H$, the sequence (x_n) defined by (8) is bounded.*

Lemma 6. *If in addition to the assumptions of Lemma 5, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\emptyset \neq A^{-1}(0) \subset F(T)$, where $T : H \to H$ is a nonexpansive map, then for any fixed $u, v_1 \in H$, the sequence (v_n) defined by (7) is bounded.*

Often, we shall use the following identity, the proof of which is well known and can easily be reproduced, see e.g, [3].

Lemma 7 (Resolvent Identity). *For any $\beta, \gamma > 0$, and $x \in H$, the identity*

$$J_\beta x = J_\gamma \left(\frac{\gamma}{\beta} x + \left(1 - \frac{\gamma}{\beta}\right) J_\beta x \right)$$

holds true.

We conclude this section with a lemma known in the literature as the subdifferential inequality. Its proof is immediate.

Lemma 8. *For all $x, y \in H$, we have*

$$\|x+y\|^2 \leq \|y\|^2 + 2\langle x, x+y \rangle.$$

3. Main results

Theorem 9. *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $\emptyset \neq A^{-1}(0) =: F \subset F(T)$, where $T : H \to H$ is a nonexpansive map. Fix $u, v_1 \in H$, and let (v_n) be the sequence generated by algorithm* (7) *with the conditions:* (i) $\alpha_n \in (0,1)$, $\lambda_n, \gamma_n \in [0,1]$, $\alpha_n + \lambda_n + \gamma_n = 1$ *with* $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$, (ii) *either (E1) or (E2),* (iii) $\beta_n \in (0,\infty)$ *with* $\liminf_{n\to\infty} \beta_n > 0$ *and either*

$$(C3) \lim_{n\to\infty} \frac{1}{\alpha_{n-1}} \left(1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n \beta_n}\right) = 0, \quad \textit{and} \quad (C4) \lim_{n\to\infty} \frac{1}{\alpha_{n-1}\alpha_n} \left(\gamma_n - \frac{\gamma_{n-1}\beta_{n+1}}{\beta_n}\right) = 0,$$

Then (v_n) converges strongly to $P_F u$, the projection of u on F.

Proof. According to Lemma 6, the sequence (v_n) is bounded. Setting

$$w_n = J_{\beta_n} \left(\alpha_{n-1} u + \lambda_{n-1} w_n + \gamma_{n-1} T w_n\right), \tag{11}$$

we see from Lemma 1 that $w_n \to P_F u$. Now it follows from (7) that

$$\begin{aligned} \|v_{n+1} - w_{n+1}\| &\leq \|v_{n+1} - w_n\| + \|w_n - w_{n+1}\| \\ &\leq \lambda_{n-1} \|v_n - w_n\| + \gamma_{n-1} \|Tv_n - Tw_n\| + \|e_{n-1}\| + \|w_n - w_{n+1}\| \\ &\leq (1-\alpha_{n-1}) \|v_n - w_n\| + \|e_{n-1}\| + \|w_n - w_{n+1}\|, \qquad (12) \end{aligned}$$

where the second inequality comes form the nonexpansivity of the map T. Using the resolvent identity, we note that (11) can be written as

$$w_n = J_\varepsilon \left(\frac{\varepsilon}{\beta_n} (\alpha_{n-1} u + \lambda_{n-1} w_n + \gamma_{n-1} T w_n) + \left(1 - \frac{\varepsilon}{\beta_n}\right) w_n \right),$$

where $\varepsilon > 0$ is the lower bound of (β_n). This together with the fact that the resolvent operator J_ε is nonexpansive gives

$$\begin{aligned}
\|w_{n+1} - w_n\| &\leq \left(1 - \frac{\varepsilon}{\beta_{n+1}}\right)\|w_{n+1} - w_n\| + \frac{\varepsilon\lambda_n}{\beta_{n+1}}\|w_{n+1} - w_n\| + \frac{\varepsilon\gamma_n}{\beta_{n+1}}\|Tw_{n+1} - Tw_n\| \\
&+ \left|\frac{\varepsilon\alpha_n}{\beta_{n+1}} - \frac{\varepsilon\alpha_{n-1}}{\beta_n}\right|\|u - w_n\| + \left|\frac{\varepsilon\gamma_n}{\beta_{n+1}} - \frac{\varepsilon\gamma_{n-1}}{\beta_n}\right|\|Tw_n - w_n\| \\
&\leq \left(1 - \frac{\varepsilon\alpha_{n-1}}{\beta_n}\right)\|w_{n+1} - w_n\| + \left|\frac{\varepsilon\alpha_n}{\beta_{n+1}} - \frac{\varepsilon\alpha_{n-1}}{\beta_n}\right|K + \left|\frac{\varepsilon\gamma_n}{\beta_{n+1}} - \frac{\varepsilon\gamma_{n-1}}{\beta_n}\right|M, \quad (13)
\end{aligned}$$

for some positive constants K and M. This last estimate reduces to

$$\|w_{n+1} - w_n\| \leq \left|1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n\beta_n}\right|K + \left|\frac{\gamma_{n-1}\beta_{n+1}}{\alpha_n\beta_n} - \frac{\gamma_n}{\alpha_n}\right|M. \quad (14)$$

Using this last inequality in (12) we arrive at

$$\begin{aligned}
\|v_{n+1} - w_{n+1}\| &\leq (1 - \alpha_{n-1})\|v_n - w_n\| + \|e_{n-1}\| + \left|1 - \frac{\alpha_{n-1}\beta_{n+1}}{\alpha_n\beta_n}\right|K \\
&+ \frac{1}{\alpha_n}\left|\gamma_n - \frac{\gamma_{n-1}\beta_{n+1}}{\beta_n}\right|M.
\end{aligned}$$

Therefore by Lemma 4, we derive $\|v_n - w_n\| \to 0$, which in turn implies that $v_n \to P_F u$. ■

Example 10. Clearly, the sequences (α_n), (β_n) and (γ_n) defined by $\alpha_n = 1/\sqrt{n+1}$, $\beta_n = 1+n^{-1}$ and $\gamma_n = 1/(n+1)$ for $n \geq 2$ satisfy the conditions (C3) and (C4).

Remark 11. A result similar to the above theorem was proved in [3] for $T = I$, the identity operator, and under the additional assumption $\beta_{n+1} \geq \alpha_n\beta_n$. Therefore, Theorem 9 is a generalization and improvement of Theorem 4 [3]. Note that Theorem 3.2 [15] which is similar in nature to Theorem 4 [3] can also be generalized in the same way.

We conclude this section by proving a strong convergence result associated with the newly introduced condition (C2). The next result is a refinement of Theorem 4 [13]. It also extends Theorem 4 [13] to general errors. Although (C2) is stronger than the condition

$$\lim_{n\to\infty}\left(\frac{1}{\beta_{n+1}} - \frac{1}{\beta_n}\right) = 0 \quad (15)$$

of Theorem 1 [4] (and Theorem 2 [4]), the main advance in Theorem 12 below is that strong convergence of the sequence (x_n) is proved under weaker conditions on both (α_n) and (λ_n) than those of Theorem 1 [4] (and Theorem 2 [4]). For the comparison of the conditions (C2) and (15), see Remark 14 below.

Theorem 12. *Assume that $A : D(A) \subset H \to 2^H$ is a maximal monotone operator and $F := A^{-1}(0) \neq \emptyset$. Fix $u, x_0 \in H$, and let (x_n) be the sequence generated by algorithm (8) with the conditions:* (i) *$\alpha_n \in (0,1)$ with $\lim_{n\to\infty}\alpha_n = 0$, and $\sum_{n=0}^{\infty}\alpha_n = \infty$,* (ii) *either (E1) or (E2),* (iii) *$\lambda_n, \gamma_n \in [0,1]$, $\alpha_n + \lambda_n + \gamma_n = 1$ with $\liminf_{n\to\infty}\gamma_n > 0$, and* (iv) *$\beta_n \in (0,\infty)$ with $\liminf_{n\to\infty}\beta_n > 0$ and (C2) being satisfied. Then (x_n) converges strongly to $P_F u$, the projection of u on F.*

Proof. From Lemma 5, we know that (x_n) is bounded. Denote

$$y_n := T_n x_n + \mu_n(u - x_n) + \sigma_n,$$

where $T_n = 2J_{\beta_n} - I$, $\mu_n = 2\alpha_n/\gamma_n$ and $\sigma_n = 2e_n/\gamma_n$. Obviously, the sequence (y_n) is bounded (since (x_n) is so), and from the definition of T_n, (5) can be written as

$$\begin{aligned} x_{n+1} &= \alpha_n u + \lambda_n x_n + \frac{\gamma_n}{2} x_n + \frac{\gamma_n}{2} T_n x_n + e_n \\ &= \left(1 - \frac{\gamma_n}{2}\right) x_n + \frac{\gamma_n}{2}\left(T_n x_n + \frac{2\alpha_n}{\gamma_n}(u - x_n) + \frac{2e_n}{\gamma_n}\right) \\ &= \left(1 - \frac{\gamma_n}{2}\right) x_n + \frac{\gamma_n}{2} y_n. \end{aligned}$$

Since T_n is nonexpansive, we have for some positive constant M

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \|T_{n+1}x_{n+1} - T_n x_n\| + \mu_{n+1}\|u - x_{n+1}\| + \mu_n \|u - x_n\| + \|\sigma_{n+1} - \sigma_n\| \\ &\leq \|T_{n+1}x_{n+1} - T_{n+1}x_n\| + \|T_{n+1}x_n - T_n x_n\| + (\mu_{n+1} + \mu_n)M + \|\sigma_{n+1} - \sigma_n\| \\ &\leq \|x_{n+1} - x_n\| + 2\left\|J_{\beta_{n+1}}x_n - J_{\beta_n}x_n\right\| + (\mu_{n+1} + \mu_n)M + \|\sigma_{n+1} - \sigma_n\| \\ &\leq \|x_{n+1} - x_n\| + 2\left\|J_{\beta_{n+1}}x_n - J_{\beta_{n+1}}\left(\frac{\beta_{n+1}}{\beta_n}x_n + \left(1 - \frac{\beta_{n+1}}{\beta_n}\right)J_{\beta_n}x_n\right)\right\| \\ &+ (\mu_{n+1} + \mu_n)M + \|\sigma_{n+1} - \sigma_n\| \\ &\leq \|x_{n+1} - x_n\| + 2\left|1 - \frac{\beta_{n+1}}{\beta_n}\right|\|x_n - J_{\beta_n}x_n\| + (\mu_{n+1} + \mu_n)M + \|\sigma_{n+1} - \sigma_n\|, \end{aligned} \tag{16}$$

where the last inequality follows from the application of the resolvent identity. Rearranging terms of (16) and passing to the limit as $n \to \infty$, we get

$$\limsup_{n\to\infty}\{\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\} \leq 0.$$

Therefore applying Lemma 2 we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0,$$

which implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{17}$$

Note that since $\liminf_{n\to\infty}\gamma_n > 0$, there exists $\delta \in [0,1)$ such that $\lambda_n \leq \delta$ for all $n \in \mathbb{N}$. Then from (8), we have

$$\begin{aligned} \|x_{n+1} - J_{\beta_n}x_n\| &\leq \alpha_n\|u - J_{\beta_n}x_n + e_n/\alpha_n\| + \lambda_n\|x_n - J_{\beta_n}x_n\| \\ &\leq \alpha_n\|u - J_{\beta_n}x_n + e_n/\alpha_n\| + \delta(\|x_n - x_{n+1}\| + \|x_{n+1} - J_{\beta_n}x_n\|), \end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|x_{n+1} - J_{\beta_n}x_n\| = 0. \tag{18}$$

On the other hand, we observe that if $\beta > 0$ is the greatest lower bound of (β_n), then the application of the resolvent identity yields

$$\begin{aligned} \|J_{\beta_n}x_n - J_\beta x_n\| &\leq \left\|\left(1 - \frac{\beta}{\beta_n}\right)(J_{\beta_n}x_n - x_n)\right\| \\ &\leq \|J_{\beta_n}x_n - x_{n+1}\| + \|x_{n+1} - x_n\|. \end{aligned}$$

Passing to the limit as $n \to \infty$ in the above inequality, and noticing (17) and (18), we have

$$\lim_{n\to\infty}\|J_{\beta_n}x_n - J_\beta x_n\| = 0. \tag{19}$$

Moreover, from (17), (18) and (19), we get

$$\lim_{n\to\infty}\|x_n - J_\beta x_n\| \leq \lim_{n\to\infty}(\|x_n - x_{n+1}\| + \|x_{n+1} - J_{\beta_n}x_n\| + \|J_{\beta_n}x_n - J_\beta x_n\|) = 0. \tag{20}$$

Now let (x_{n_k}) be a subsequence of (x_n) converging weakly to some z. Then for some $K > 0$,

$$\begin{aligned} 2\langle x_{n_k} - J_\beta z, z - J_\beta z\rangle &= \|x_{n_k} - J_\beta z\|^2 + \|z - J_\beta z\|^2 - \|x_{n_k} - z\|^2 \\ &\leq (\|x_{n_k} - J_\beta x_{n_k}\| + \|x_{n_k} - z\|)^2 + \|z - J_\beta z\|^2 \\ &- \|x_{n_k} - z\|^2 \\ &\leq K\|x_{n_k} - J_\beta x_{n_k}\| + \|z - J_\beta z\|^2. \end{aligned}$$

Passing to the limit in the above inequality as $k \to \infty$, and noticing (20), we arrive at $z \in A^{-1}(0)$. Hence for some subsequence (x_{n_j}) of (x_n) converging weakly to a point x_∞, say, we have

$$\limsup_{n\to\infty}\langle u - P_F u, x_n - P_F u\rangle = \lim_{j\to\infty}\langle u - P_F u, x_{n_j} - P_F u\rangle = \langle u - P_F u, x_\infty - P_F u\rangle \leq 0.$$

Finally, from Lemma 8 and (8), we have

$$\begin{aligned} \|x_{n+1} - P_F u\| &\leq (\lambda_n \|x_n - P_F u\| + \gamma_n \|J_{\beta_n} x_n - P_F u\|)^2 + 2\alpha_n \left\langle u - P_F u + \frac{e_n}{\alpha_n}, x_{n+1} - P_F u\right\rangle \\ &\leq (1 - \alpha_n)\|x_n - P_F u\|^2 + 2\alpha_n \left\langle u - P_F u + \frac{e_n}{\alpha_n}, x_{n+1} - P_F u\right\rangle. \end{aligned}$$

Therefore, from Lemma 4 we derive strong convergence of (x_n) to $P_F u$. In the case when the error sequence (e_n) satisfy condition (E1), then we get from Lemma 8 and (8)

$$\|x_{n+1} - P_F u\| \leq (1 - \alpha_n)\|x_n - P_F u\|^2 + 2\alpha_n \langle u - P_F u, x_{n+1} - P_F u\rangle + \|e_n\| C,$$

for some positive constant C. As before, strong convergence of (x_n) to $P_F u$ can be derived. ∎

Remark 13. Unlike in Theorem 4 [4] and Theorem 3.3 [16], we do not require γ_n to be bounded above away from 1. In addition, we have used the weaker condition (C2) instead of (C1). Therefore, Theorem 12 improves significantly the results of [4] and [16].

Remark 14. Note that for the sequence (β_n) satisfying $\beta_n \geq \varepsilon$ for some $\varepsilon > 0$ and all $n \in \mathbb{N}$,

$$\left|\frac{1}{\beta_{n+1}} - \frac{1}{\beta_n}\right| = \frac{1}{\beta_{n+1}}\left|1 - \frac{\beta_{n+1}}{\beta_n}\right| \leq \frac{1}{\varepsilon}\left|1 - \frac{\beta_{n+1}}{\beta_n}\right|$$

implying that the condition (15) is weaker than the condition (C2) of the preceding theorem. Indeed, one can very that the sequence

$$\beta_n = \begin{cases} 2n & \text{if } n \text{ is odd,} \\ 3n & \text{if } n \text{ is even} \end{cases}$$

satisfy (15) but not (C2). These two conditions are however equivalent if (β_n) is bounded (both from below away from zero and from above).

References

[1] O. A. Boikanyo and G. Moroşanu, *Modified Rockafellar's algorithms*, Math. Sci. Res. J. (5) 13 (2009), 101-122.

[2] ———, *A proximal point algorithm converging strongly for general errors*, Optim. Lett., 4 (2010), 635-641.

[3] ———, *Inexact Halpern-type proximal point algorithms*, J. Glob. Optim., in press.

[4] ———, *Four parameter proximal point algorithms*, Nonlinear Analysis, 74 (2011) 544-555.

[5] K. Gobel and W. A. Kirk, *Topics on Metric Fixed Point Theory*, Cambridge University Press, Cambridge, (1990).

[6] O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim. 29 (1991), 403-419.

[7] S. Kamimura and W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory 106 (2000), 226-240.

[8] N. Lehdili and A. Moudafi, *Combining the proximal algorithm and Tikhonov regularization,* Optimization 37 (1996), 239-252.

[9] B. Martinet, *Régularisation d'inéquations variationnelles par approximations successives*, Rev. Française Informat. Recherche Opérationnelle 4 (1970), Ser. R-3, 154–158.

[10] R. T. Rockafellar, *Monotone operators and the proximal point algorithm,* SIAM J. Control Optim. 14 (1976), 877-898.

[11] M. V. Solodov and B. F. Svaiter, *Forcing strong convergece of proximal point iterations in a Hilbert space,* Math. Program. Ser. A 14 (2000), 189-202.
[12] T. Suzuki, *Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces*, Fixed Point Theory Appl., (2005), 103-123.
[13] F. Wang, *A note on the regularized proximal point algorithm,* J. Glob. Optim., in press.
[14] H. K. Xu, *Iterative algorithms for nonlinear operators,* J. London Math. Soc. (2) 66 (2002), 240-256.
[15] ______, *A regularization method for the proximal point algorithm,* J. Glob. Optim. 36 (2006), 115-125.
[16] Y. Yao and M. A. Noor, *On convergence criteria of generalized proximal point algorithms*, J. Comp. Appl. Math., 217 (2008), 46-55.

Department of Mathematics and its Applications, Central European University, Nador u. 9, H-1051 Budapest, Hungary
E-mail address: boikanyo_oganeditse@ceu-budapest.edu, and morosanug@ceu.hu