

FROM METRIC PROPERTIES TO A RIGIDITY CONJECTURE ON FINSLER MANIFOLDS VIA A GEODESIC DETECTING ALGORITHM

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Abstract. The aim of this paper is to consider Busemann-type inequalities on Finsler manifolds. We actually formulate a rigidity conjecture: any Finsler manifold which is a Busemann NPC space is Berwaldian. This statement is supported by some theoretical results and numerical examples. The presented examples are obtained by using evolutionary techniques (genetic algorithms) which can be used for detecting geodesics on a large class of not necessarily reversible Finsler manifolds.

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1 Introduction

In the forties, Busemann developed a synthetic geometry of G -spaces, see [3]. These spaces possess the essential geometric properties of Finsler manifolds but they have no necessarily a differential structure. Nowadays, Busemann's curvature is referred to non-positively curved (quasi-)metric spaces, called *Busemann NPC spaces*; this notion of non-positive curvature requires that in small geodesic triangles the length of a side is at least the twice of the geodesic distance of the mid-points of the other two sides, see [3, p. 237]. Note that on Riemannian manifolds the non-positivity of the sectional curvature characterizes Busemann's non-positivity curvature.

In the fifties, Aleksandrov introduced independently another notion of curvature on metric spaces, a stronger one than that of Busemann, see Jost [5, Section 2.3]. Nevertheless, on Riemannian manifolds the Aleksandrov curvature condition holds if and only if the sectional curvature is non-positive, see Bridson-Haefliger [2, Theorem 1A.6]. In the Finsler case the picture is quite *rigid*: if on a reversible Finsler manifold (M, F) the Aleksandrov curvature condition holds (on the induced metric space by (M, F)) then (M, F) should be Riemannian, see [2, Proposition 1.14].

Having these facts in our mind, a natural question arises: *What about Finsler manifolds that are Busemann NPC spaces?*

Clearly, we are thinking first to those Finsler manifolds which have non-positive flag curvature. However, within this class, we easily find examples which are not Busemann NPC spaces. For instance, the Hilbert metric of the interior of a simple, closed and strictly convex curve \mathbf{c} in the Euclidean plane defines a Finsler metric with constant flag curvature -1 , but Busemann's curvature condition holds if and only if \mathbf{c} is an ellipse, see Kelly-Straus [6]. In the latter case, the Hilbert metric becomes a Riemannian one.

The above example suggests we face again a rigidity result as in the Aleksandrov case. However, a recent result of Kozma-Kristály-Varga [8] entitle us to believe that the appropriate class of Finsler manifolds is the *Berwald* one, a larger class than Riemannian manifolds. Namely, we formulate the following

Conjecture. *Let (M, F) be a (not necessarily reversible) Finsler manifold such that (M, d_F) is a Busemann NPC space. Then (M, F) is a Berwald space.*

The aim of the present paper is to recall some theoretical developments and certain numerical examples in order to support the above Conjecture. In addition, we propose a new numerical approach to detect minimal geodesics on Finsler manifolds by using evolutionary techniques. Note that the presented numerical approach can be efficiently used for verifying different metric relations on a large class of Finsler manifolds exploiting the basic property of Finslerian-geodesics that they locally minimize the integral length.

The paper is organized as follows. In the next section we recall the notion of Busemann NPC (quasi-)metric spaces and basic elements of Finsler manifolds. Section 3 deals with the Conjecture from both theoretical and numerical point of view. Section 4 is independent from the previous ones and describes the detection of geodesics on Finsler manifolds by using genetic algorithms.

2 Preliminaries

2.1 Busemann NPC spaces

Let (M, d) be a quasi-metric space (i.e., d is not necessarily symmetric). A continuous curve $\mathbf{c} : [0, r] \rightarrow M$ with $\mathbf{c}(0) = \mathbf{p}$, $\mathbf{c}(r) = \mathbf{q}$ is a *shortest geodesic*, if $l(\mathbf{c}) = d(\mathbf{p}, \mathbf{q})$, where $l(\mathbf{c})$ is the *generalized length* of \mathbf{c} and it is defined by

$$l(\mathbf{c}) = \sup \left\{ \sum_{i=1}^n d(\mathbf{c}(t_{i-1}), \mathbf{c}(t_i)) : 0 = t_0 < t_1 < \dots < t_n = r, n \in \mathbb{N} \right\}.$$

We say that (M, d) is a *locally geodesic (length) space* if for every point $\mathbf{p} \in M$ there is a neighborhood $N_{\mathbf{p}} \subset M$ such that for every two points $\mathbf{x}, \mathbf{y} \in N_{\mathbf{p}}$ there exists a shortest geodesic joining them. In the sequel, we always assume that the shortest geodesics are parametrized proportionally to the arc length, i.e., $l(\mathbf{c}|_{[0,t]}) = t \cdot l(\mathbf{c}), \forall t \in [0, r]$.

Definition 1 (Busemann NPC space) *A locally geodesic space (M, d) is said to be a Busemann non-positive curvature space (shortly, Busemann NPC space), if for every $\mathbf{p} \in M$ there exists a neighborhood $N_{\mathbf{p}} \subset M$ such that for any two shortest geodesics $\mathbf{c}_1, \mathbf{c}_2 : [0, 1] \rightarrow M$ with $\mathbf{c}_1(0) = \mathbf{c}_2(0) \in N_{\mathbf{p}}$ and with endpoints $\mathbf{c}_1(1), \mathbf{c}_2(1) \in N_{\mathbf{p}}$ we have*

$$2d\left(\mathbf{c}_1\left(\frac{1}{2}\right), \mathbf{c}_2\left(\frac{1}{2}\right)\right) \leq d(\mathbf{c}_1(1), \mathbf{c}_2(1)). \quad (1)$$

2.2 Finsler manifolds

Let M be a connected m -dimensional C^∞ manifold and let $TM = \bigcup_{\mathbf{p} \in M} T_{\mathbf{p}}M$ be its tangent bundle. If the continuous function $F : TM \rightarrow [0, \infty)$ satisfies the conditions that it is C^∞ on $TM \setminus \{\mathbf{0}\}$; $F(\mathbf{p}, t\mathbf{y}) = tF(\mathbf{p}, \mathbf{y})$ for all $t \geq 0$ and $\mathbf{y} \in T_{\mathbf{p}}M$, i.e., F is positively homogeneous of degree one; and the matrix $g_{ij}(\mathbf{y}) := \left[\frac{1}{2}F^2\right]_{y^i y^j}(\mathbf{y})$ is positive definite for all $\mathbf{y} \in TM \setminus \{\mathbf{0}\}$, then we say that (M, F) is a *Finsler manifold*. If F is absolutely homogeneous, then (M, F) is said to be *reversible*.

Let π^*TM be the pull-back of the tangent bundle TM by $\pi : TM \setminus \{\mathbf{0}\} \rightarrow M$. Unlike the Levi-Civita connection in Riemann geometry, there is no unique natural connection in the Finsler case. Among these connections on π^*TM , we choose the *Chern connection* whose coefficients are denoted by Γ_{ij}^k ; see Bao-Chern-Shen [1, p. 38]. This connection induces the *curvature tensor*, denoted by R ; see [1, Chapter 3]. The Chern connection defines the *covariant derivative* $D_{\mathbf{V}}\mathbf{U}$ of a vector field \mathbf{U} in the direction $\mathbf{V} \in T_{\mathbf{p}}M$. Since, in general, the Chern connection coefficients Γ_{jk}^i in natural coordinates have a directional dependence, we must say explicitly that $D_{\mathbf{V}}\mathbf{U}$ is defined with a fixed reference vector. In particular, let $\mathbf{c} : [0, r] \rightarrow M$ be a smooth curve with velocity field $\mathbf{T} = \mathbf{T}(t) = \dot{\mathbf{c}}(t) = (T^k)_{k=1,m}$. Suppose that $\mathbf{U} = (U^i)_{i=1,m}$ and \mathbf{W} are vector fields defined along \mathbf{c} . We define $D_{\mathbf{T}}\mathbf{U}$ with *reference vector* \mathbf{W} as

$$D_{\mathbf{T}}\mathbf{U} = \left[\frac{dU^i}{dt} + U^j T^k (\Gamma_{jk}^i)_{(\mathbf{c}, \mathbf{W})} \right] \frac{\partial}{\partial p^i} \Big|_{\mathbf{c}(t)},$$

where $\left\{ \frac{\partial}{\partial p^i} \Big|_{\mathbf{c}(t)} \right\}_{i=1,m}$ is a basis of $T_{\mathbf{c}(t)}M$. A C^∞ curve $\mathbf{c} : [0, r] \rightarrow M$, with velocity $\mathbf{T} = \dot{\mathbf{c}}$ is a (Finslerian) *geodesic* if

$$D_{\mathbf{T}} \left[\frac{\mathbf{T}}{F(\mathbf{T})} \right] = 0 \quad \text{with reference vector } \mathbf{T}. \quad (2)$$

If the Finslerian speed of the geodesic \mathbf{c} is constant, then (2) becomes

$$\frac{d^2 \mathbf{c}^i}{dt^2} + \frac{d\mathbf{c}^j}{dt} \frac{d\mathbf{c}^k}{dt} (\Gamma_{jk}^i)_{(\mathbf{c}, \mathbf{T})} = 0, \quad i = 1, \dots, m = \dim M. \quad (3)$$

In the sequel, we assume that the geodesics have constant Finslerian speed.

A Finsler manifold (M, F) is said to be *forward* (resp. *backward*) *geodesically complete* if every geodesic $\mathbf{c} : [0, 1] \rightarrow M$ can be extended to a geodesic defined on $[0, \infty)$ (resp. $(-\infty, 1]$).

Let $\mathbf{c} : [0, r] \rightarrow M$ be a piecewise C^∞ curve. Its *integral length* is defined as

$$L_F(\mathbf{c}) = \int_0^r F(\mathbf{c}(t), \dot{\mathbf{c}}(t)) dt.$$

For $\mathbf{p}, \mathbf{q} \in M$, denote by $\Gamma_{[0, r]}(\mathbf{p}, \mathbf{q})$ the set of all piecewise C^∞ curves $\mathbf{c} : [0, r] \rightarrow M$ such that $\mathbf{c}(0) = \mathbf{p}$ and $\mathbf{c}(r) = \mathbf{q}$. Define the map $d_F : M \times M \rightarrow [0, \infty)$ by

$$d_F(\mathbf{p}, \mathbf{q}) = \inf_{\mathbf{c} \in \Gamma_{[0, r]}(\mathbf{p}, \mathbf{q})} L_F(\mathbf{c}). \quad (4)$$

Of course, we have $d_F(\mathbf{p}, \mathbf{q}) \geq 0$, where equality holds if and only if $\mathbf{p} = \mathbf{q}$, and the *triangle inequality* holds, i.e., $d_F(\mathbf{p}_0, \mathbf{p}_2) \leq d_F(\mathbf{p}_0, \mathbf{p}_1) + d_F(\mathbf{p}_1, \mathbf{p}_2)$ for every $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2 \in M$. In general, since F is only a positive homogeneous function, $d_F(\mathbf{p}, \mathbf{q}) \neq d_F(\mathbf{q}, \mathbf{p})$; thus, (M, d_F) is only a quasi-metric space. If (M, g) is a Riemannian manifold, we will use the notation d_g instead of d_F which becomes a usual metric function.

Busemann and Mayer [4, Theorem 2, p. 186] proved that the generalized length $l(\mathbf{c})$ and the integral length $L_F(\mathbf{c})$ of any (piecewise) C^∞ curve coincide for Finsler manifolds.

Let $(\mathbf{p}, \mathbf{y}) \in TM \setminus \{\mathbf{0}\}$ and let \mathbf{V} be a section of the pulled-back bundle π^*TM . Then,

$$K(\mathbf{y}, \mathbf{V}) = \frac{g_{(\mathbf{p}, \mathbf{y})}(R(\mathbf{V}, \mathbf{y})\mathbf{y}, \mathbf{V})}{g_{(\mathbf{p}, \mathbf{y})}(\mathbf{y}, \mathbf{y})g_{(\mathbf{p}, \mathbf{y})}(\mathbf{V}, \mathbf{V}) - [g_{(\mathbf{p}, \mathbf{y})}(\mathbf{y}, \mathbf{V})]^2} \quad (5)$$

is the *flag curvature* with flag \mathbf{y} and transverse edge \mathbf{V} . Here,

$$g_{(\mathbf{p}, \mathbf{y})} := g_{ij(\mathbf{p}, \mathbf{y})} dp^i \otimes dp^j := \left[\frac{1}{2} F^2 \right]_{y^i y^j} dp^i \otimes dp^j, \quad \mathbf{p} \in M, \mathbf{y} \in T_{\mathbf{p}}M, \quad (6)$$

is the Riemannian metric on the pulled-back bundle π^*TM . In particular, when the Finsler structure F arises from a Riemannian metric g (i.e., the fundamental tensor $g_{ij} = \left[\frac{1}{2} F^2 \right]_{y^i y^j}$ does not depend on the direction \mathbf{y}), the flag curvature coincides with the usual *sectional curvature*.

If $K(\mathbf{V}, \mathbf{W}) \leq 0$ for every $\mathbf{0} \neq \mathbf{V}, \mathbf{W} \in T_{\mathbf{p}}M$, and $\mathbf{p} \in M$, with \mathbf{V} and \mathbf{W} not collinear, we say that the flag curvature of (M, F) is *non-positive*.

A Finsler manifold is of *Berwald type* if the Chern connection coefficients Γ_{ij}^k in natural coordinates depend only on the base point. Special Berwald spaces are the (locally) Minkowski spaces and the Riemannian manifolds.

3 Busemann NPC spaces versus Finsler manifolds: the Conjecture

3.1 Theoretical results

Let (M, g) be a Riemannian manifold and d_g the metric function induced by g . In the fifties, Busemann proved the following result.

Theorem 2 ([3, Theorem (41.6)]) *Let (M, g) be a Riemannian manifold. The metric space (M, d_g) is a Busemann NPC space if and only if the sectional curvature of (M, g) is non-positive.*

Due to Theorem 2, Shen [12] formulated an open problem on the existence of a curvature notion \mathcal{B} on a Finsler manifold (M, F) with the property that $\mathcal{B} \leq 0$ if and only if (M, d_F) is a Busemann NPC space. For Finsler manifolds, as far as we know, the unique result concerning Busemann-type inequalities is due to Kozma, Kristály and Varga (cf. [8] and [7]).

Theorem 3 *Let (M, F) be a (not necessarily reversible) Berwald space with non-positive flag curvature. Then (M, d_F) is a Busemann NPC space.*

Remark 4 Beside Riemannian manifolds, Theorem 3 easily follows for (not necessarily reversible) Minkowski spaces. Indeed, since M is a vector space and F is a Minkowski norm inducing a Finsler structure on M by translation, its flag curvature is identically zero, the geodesics are straight lines, and in the Busemann-type inequality (see Definition 1) we always have equality.

Note that there are Berwald spaces with non-positive flag curvature which are neither Riemannian manifolds nor Minkowski spaces. In the sequel, we give such an example. Let (N, h) be an arbitrarily closed hyperbolic Riemannian manifold of dimension at least 2, and $\varepsilon > 0$. Let us define the Finsler metric $F_\varepsilon : T(\mathbb{R} \times N) \rightarrow [0, \infty)$ by

$$F_\varepsilon(t, \mathbf{p}; \tau, \mathbf{w}) = \sqrt{h_{\mathbf{p}}(\mathbf{w}, \mathbf{w}) + \tau^2 + \varepsilon \sqrt{h_{\mathbf{p}}^2(\mathbf{w}, \mathbf{w}) + \tau^4}},$$

where $(t, \mathbf{p}) \in \mathbb{R} \times N$ and $(\tau, \mathbf{w}) \in T_{(t, \mathbf{p})}(\mathbb{R} \times N)$. Shen [9] pointed out that the pair $(\mathbb{R} \times N, F_\varepsilon)$ is a reversible Berwald space with non-positive flag curvature.

The proof of Theorem 3 exploits the fact that the Finsler manifold we are dealing with is actually a *Berwald space*: (a) the reverse of a given geodesic segment is also a geodesic (which is not true for every non-reversible Finsler manifold); (b) the reference vectors are not relevant since the Chern connection coefficients do not depend on the direction.

We conclude this subsection with the following simple observation which is based on the Cartan-Hadamard theorem [1, p. 238], see also [5, Corollary 2.2.4].

Proposition 5 *Let (M, F) be a (not necessarily reversible) forward or backward geodesically complete, simply connected Finsler manifold such that (M, d_F) is a Busemann NPC space. Then (M, d_F) is a global Busemann NPC space, i.e., inequality (1) holds for any pair of geodesics.*

3.2 Supporting the Conjecture: Examples

In the sequel we provide some relevant examples which support the Conjecture beside the theoretical results presented in the previous subsection.

1. Hilbert geometry. We consider the Hilbert metric of the interior of a simple, closed and strictly convex curve \mathbf{c} in the Euclidean plane. In order to describe this metric, let $M_{\mathbf{c}} \subset \mathbb{R}^2$ be the region defined by the interior of the curve \mathbf{c} and fix $\mathbf{p}_1, \mathbf{p}_2 \in M_{\mathbf{c}}$. Assume first that $\mathbf{p}_1 \neq \mathbf{p}_2$. Since \mathbf{c} is a convex curve, the straight line passing to the points $\mathbf{p}_1, \mathbf{p}_2$ intersects the curve \mathbf{c} in two points; denote them by $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{c}$. Then, there are $\tau_1, \tau_2 \in (0, 1)$ such that $\mathbf{p}_i = \tau_i \mathbf{u}_1 + (1 - \tau_i) \mathbf{u}_2$ ($i = 1, 2$). The *Hilbert distance* between \mathbf{p}_1 and \mathbf{p}_2 is

$$d_H(\mathbf{p}_1, \mathbf{p}_2) = \left| \log \left(\frac{1 - \tau_1}{1 - \tau_2} \cdot \frac{\tau_2}{\tau_1} \right) \right|.$$

We complete this definition by $d_H(\mathbf{p}, \mathbf{p}) = 0$ for every $\mathbf{p} \in M_{\mathbf{c}}$. One can easily prove that $(M_{\mathbf{c}}, d_H)$ is a metric space and defines a reversible, projective Finsler metric

$$F_H(\mathbf{p}, \mathbf{y}) = \lim_{t \rightarrow 0^+} \frac{d_H(\mathbf{p}, \mathbf{p} + t\mathbf{y})}{t}$$

with constant flag curvature -1 . However, due to Kelly-Straus, we have

Proposition 6 (see [6]) *The metric space $(M_{\mathbf{c}}, d_H)$ is a Busemann NPC space if and only if the curve $\mathbf{c} \subset \mathbb{R}^2$ is an ellipse.*

Consequently, $(M_{\mathbf{c}}, d_H)$ is a Busemann NPC metric space if and only if $(M_{\mathbf{c}}, F_H)$ is a Riemannian manifold.

2. A projectively flat Finsler metric with negative flag curvature. Fix $\varepsilon \in [-1, 1)$. For any $\mathbf{p} \in B^m(\mathbf{0}, 1) = \{\mathbf{p} \in \mathbb{R}^m : |\mathbf{p}| < 1\}$ and $\mathbf{y} \in T_{\mathbf{p}}B^m(\mathbf{0}, 1) = \mathbb{R}^m$ we define

$$F_\varepsilon(\mathbf{p}, \mathbf{y}) = \frac{1}{2} \left\{ \frac{\sqrt{|\mathbf{y}|^2 - (|\mathbf{p}|^2 |\mathbf{y}|^2 - \langle \mathbf{p}, \mathbf{y} \rangle^2)} + \langle \mathbf{p}, \mathbf{y} \rangle}{1 - |\mathbf{p}|^2} - \frac{\varepsilon \sqrt{|\mathbf{y}|^2 - \varepsilon^2 (|\mathbf{p}|^2 |\mathbf{y}|^2 - \langle \mathbf{p}, \mathbf{y} \rangle^2)} + \varepsilon^2 \langle \mathbf{p}, \mathbf{y} \rangle}{1 - \varepsilon^2 |\mathbf{p}|^2} \right\}. \quad (7)$$

The pair $(B^m(\mathbf{0}, 1), F_\varepsilon)$ is a forward geodesically complete Finsler manifold for any $\varepsilon \in (-1, 1)$. When $\varepsilon = -1$, F_{-1} is the well-known Klein/Hilbert metric on $B^m(\mathbf{0}, 1)$ and $(B^m(\mathbf{0}, 1), d_{F_{-1}})$ is a global Busemann NPC space, see also the previous example for $m = 2$. Note that F_ε ($\varepsilon \in [-1, 1)$) is projectively flat with constant flag curvature -1 , see Shen [11, p. 1722]. When $\varepsilon \in (-1, 1)$, $(B^m(\mathbf{0}, 1), F_\varepsilon)$ is not Berwaldian. On account of Proposition 5, if $(B^m(\mathbf{0}, 1), d_{F_\varepsilon})$ is a Busemann NPC space, the quasi-metric space $(B^m(\mathbf{0}, 1), d_{F_\varepsilon})$ is a global Busemann NPC space. However, Figure 1 (a) shows that one can easily find (large) geodesic triangles where Busemann-type inequality fails. This fact does not imply that Busemann-type inequality cannot hold in particular, see Figure 1 (b).

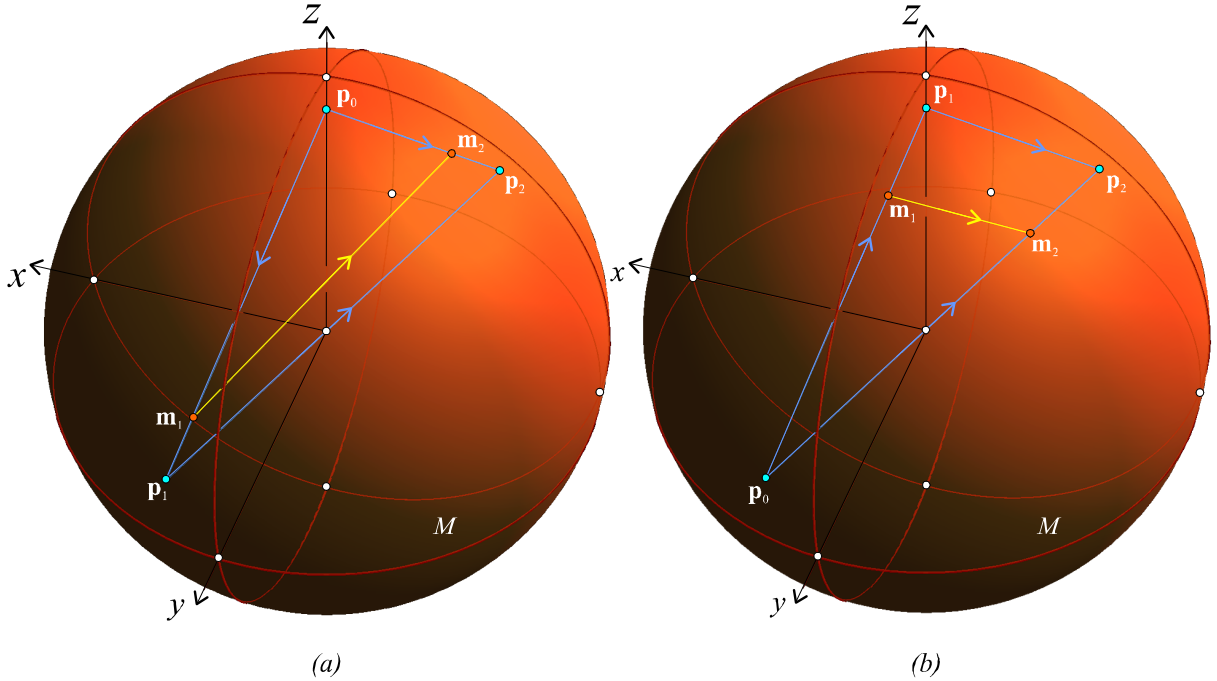


Figure 1: Testing the Busemann-type inequality in case of the Finsler metric (7) for $\varepsilon = 0.1$. In case of the geodesic triangle (a) the Busemann-type inequality does not hold. The nodes of the triangle are $\mathbf{p}_0(0.0, 0.0, 0.9)$, $\mathbf{p}_1(0.5, 0.5, -0.5)$ and $\mathbf{p}_2(-0.5, -0.5, 0.5)$. The lengths of the sides and of the median are $d_{F_\varepsilon}(\mathbf{p}_0, \mathbf{p}_1) = 1.14416728$, $d_{F_\varepsilon}(\mathbf{p}_0, \mathbf{p}_2) = 0.679288454$, $d_{F_\varepsilon}(\mathbf{p}_1, \mathbf{p}_2) = 1.23015501$ and $d_{F_\varepsilon}(\mathbf{m}_1, \mathbf{m}_2) = 0.921800308$, respectively. However, in case of the geodesic triangle (b) the Busemann-type inequality holds. The nodes of this triangle were obtained from triangle (a) by interchanging the nodes \mathbf{p}_0 and \mathbf{p}_1 . In this case we have $d_{F_\varepsilon}(\mathbf{p}_0, \mathbf{p}_1) = 1.28109422$, $d_{F_\varepsilon}(\mathbf{p}_0, \mathbf{p}_2) = 1.23015501$, $d_{F_\varepsilon}(\mathbf{p}_1, \mathbf{p}_2) = 0.679288454$ and $d_{F_\varepsilon}(\mathbf{m}_1, \mathbf{m}_2) = 0.312663329$.

3. Finslerian-Poincaré disc. Let us consider the disc

$$M = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 4\}.$$

Introducing the polar coordinates (r, θ) on M , i.e., $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, we define the non-reversible Finsler metric on M by

$$F((r, \theta), \mathbf{V}) = \frac{1}{1 - \frac{r^2}{4}} \sqrt{p^2 + r^2 q^2} + \frac{pr}{1 - \frac{r^4}{16}},$$

where

$$\mathbf{V} = p \frac{\partial}{\partial r} + q \frac{\partial}{\partial \theta} \in T_{(r, \theta)} M.$$

The pair (M, F) is the so-called *Finslerian-Poincaré disc*. Within the classification of Finsler manifolds, (M, F) is a *Randers space*, see [1, Section 12.6], which is a forward (but not backward) geodesically complete manifold having constant negative flag curvature $-1/4$. Due to Szabó's rigidity theorem about Berwald surfaces (see for instance [1, Theorem 10.6.2]), (M, F) is *not* a Berwald space. If (M, d_F) is a Busemann NPC space, due to Proposition 5, the quasi-metric space (M, d_F) is a global Busemann NPC space. However, Figure 2 shows that one can find geodesic triangles where Busemann-type inequalities do not hold.

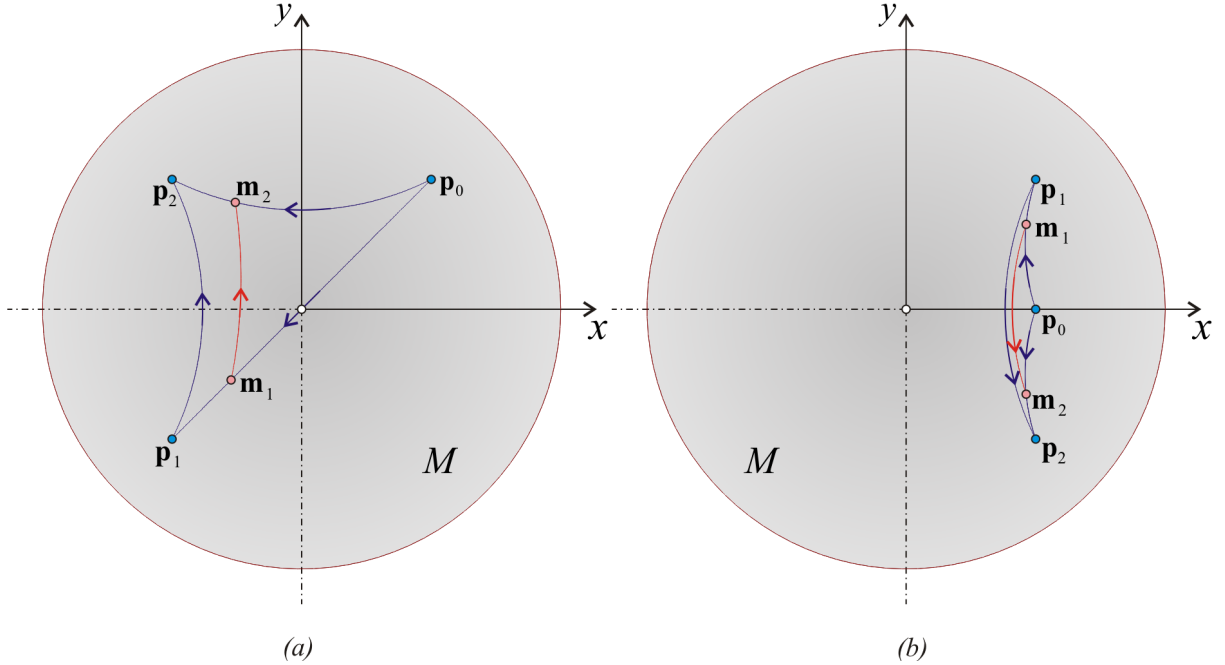


Figure 2: Two geodesic triangles in Finslerian-Poincaré disc from many others for which the Busemann-type inequality fails. (a) The nodes of the geodesic triangle are $\mathbf{p}_0 (1, 1)$, $\mathbf{p}_1 (-1, -1)$ and $\mathbf{p}_2 (-1, 1)$. The lengths of the sides and of the median are $d_F(\mathbf{p}_0, \mathbf{p}_1) = 3.5254963$, $d_F(\mathbf{p}_0, \mathbf{p}_2) = 2.88728397$, $d_F(\mathbf{p}_1, \mathbf{p}_2) = 2.88728399$ and $d_F(\mathbf{m}_1, \mathbf{m}_2) = 1.71860536$, respectively. (b) Another geodesic triangle the nodes of which are $\mathbf{p}_0 (1, 0)$, $\mathbf{p}_1 (1, 1)$ and $\mathbf{p}_2 (1, -1)$. In this case we have $d_F(\mathbf{p}_0, \mathbf{p}_1) = 2.07878347$, $d_F(\mathbf{p}_0, \mathbf{p}_2) = 2.07878333$, $d_F(\mathbf{p}_1, \mathbf{p}_2) = 2.88728393$ and $d_F(\mathbf{m}_1, \mathbf{m}_2) = 1.70407783$.

4. A non-projectively flat Finsler metric with zero flag curvature. Let $m \geq 2$ and consider the cylinder

$$M = \{\mathbf{p} = ((x_1, x_2), \tilde{\mathbf{x}}_3) \in \mathbb{R}^2 \times \mathbb{R}^{m-2} : x_1^2 + x_2^2 < 1\}.$$

For any $\mathbf{y} = ((y_1, y_2), \tilde{\mathbf{y}}_3) \in T_{\mathbf{p}}M = \mathbb{R}^m$ we define

$$F(\mathbf{p}, \mathbf{y}) = \frac{\sqrt{(-x_2y_1 + x_1y_2)^2 + |\mathbf{y}|^2(1 - x_1^2 - x_2^2)} - (-x_2y_1 + x_1y_2)}{1 - x_1^2 - x_2^2}. \quad (8)$$

The pair (M, F) is an m -dimensional Finsler manifold with constant flag curvature 0, see Shen [10, Theorem 1.1] and [9]. Moreover, (M, F) is *not* Berwaldian and it is not forward geodesically complete. For $m = 2$ and $m = 3$, we present some constellations of small geodesic triangles where Busemann-type inequality does not hold, see on Figure 3 (a) and (b), respectively. Due to the lack of geodesically completeness, we cannot apply Proposition 5. Therefore, we have to consider "small" geodesic triangles which are inside of a normal neighborhood of some point $\mathbf{p} \in M$ (see the set $N_{\mathbf{p}}$ from Definition 1). At the moment we cannot estimate the size of the normal neighborhood $N_{\mathbf{p}}$ for a fixed point $\mathbf{p} \in M$.

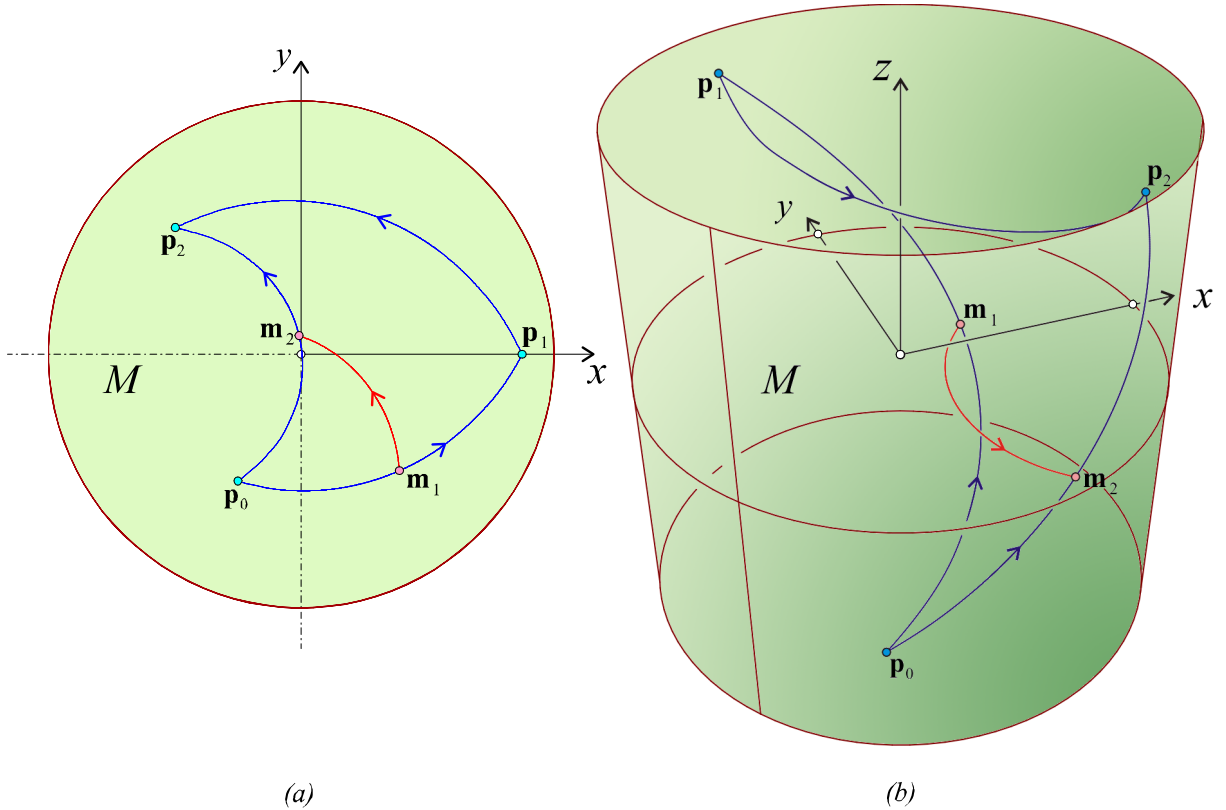


Figure 3: Two geodesic triangles are presented from many others for which the Busemann-type inequality fails in the 2- and 3-dimensional cases of the Finsler metric (8). (a) The nodes of the geodesic triangle are $\mathbf{p}_0 (-0.25, -0.5)$, $\mathbf{p}_1 (0.875, 0)$ and $\mathbf{p}_2 (-0.4975, 0.5)$. The lengths of the sides and of the median are $d_F(\mathbf{p}_0, \mathbf{p}_1) = 0.845022533$, $d_F(\mathbf{p}_0, \mathbf{p}_2) = 1.2647537$, $d_F(\mathbf{p}_1, \mathbf{p}_2) = 0.999184207$ and $d_F(\mathbf{m}_1, \mathbf{m}_2) = 0.629171204$, respectively. (b) Another geodesic triangle the nodes of which are $\mathbf{p}_0 (-0.25, -0.5, -1.0)$, $\mathbf{p}_1 (-0.4975, 0.5, 1.0)$ and $\mathbf{p}_2 (0.995, 0.0, 0.5)$. In this case we have $d_F(\mathbf{p}_0, \mathbf{p}_1) = 2.28325263$, $d_F(\mathbf{p}_0, \mathbf{p}_2) = 1.59552717$, $d_F(\mathbf{p}_1, \mathbf{p}_2) = 1.63538395$ and $d_F(\mathbf{m}_1, \mathbf{m}_2) = 1.03707673$.

4 Finding Finslerian-geodesics via evolutionary techniques

Evolutionary techniques are useful and efficient in case of such optimization problems when the search space is large, complex or traditional search and numerical methods fail. They are based on principles of the evolution via natural *selection*, employing a population of individuals that undergo selection in the presence of variation-inducing *operators* such as *mutation* and *recombination*.

Consider the m -dimensional ($m \geq 2$) Finsler manifold (M, F) and the set of curves

$$\Gamma_{[0,1]}(\mathbf{p}, \mathbf{q}) = \{\mathbf{c} \in C^{k-2}([0, 1], M) : \mathbf{c}(0) = \mathbf{p}, \mathbf{c}(1) = \mathbf{q}\},$$

where the order $k - 2$ of continuity is at least 1 and the points $\mathbf{p}, \mathbf{q} \in M$ are preliminary fixed. At the moment, the proposed algorithm provides results when $M \subset \mathbb{R}^m$.

A genetic algorithm is presented as a computer simulation in which isolated populations of candidate solutions to the optimization problem

$$\begin{cases} L_F(\mathbf{c}) = \int_0^1 F(\mathbf{c}(t), \dot{\mathbf{c}}(t)) dt \rightarrow \min \\ \mathbf{c} \in \Gamma_{[0,1]}(\mathbf{p}, \mathbf{q}) \end{cases} \quad (9)$$

evolve towards better solutions. From Finsler geometrical point of view we exploit the fact that short geodesics minimize the integral length, see [1, Theorem 6.3.1]. Thus, in the sequel we implicitly assume that \mathbf{p} and \mathbf{q} are close enough to each other in the sense that they belong to a common normal neighborhood $N_{\tilde{\mathbf{p}}}$ of a certain point $\tilde{\mathbf{p}} \in M$.

In fact the algorithm is a generational process (controlled by selection, recombination/crossover and mutation) that is repeated until a termination condition has been reached. Common terminating conditions are: a solution is found that satisfies minimum criteria, a fixed number of generations is reached, the fitness of the highest ranking solution is at a such a level that successive iterations no longer produce better results, etc.

Before starting the genetic representation of the individuals that are involved in the search processes, we recall the definition and some properties of B-spline curves that will be used in our further investigations.

Definition 7 (Normalized B-spline basis functions) *The recursive function $N_j^k(t)$ ($k \geq 1$) given by the equations*

$$N_j^1(t) = \begin{cases} 1, & t \in [t_j, t_{j+1}) \\ 0, & \text{otherwise} \end{cases}$$

$$N_j^k(t) = \frac{t - t_j}{t_{j+k-1} - t_j} N_j^{k-1}(t) + \frac{t_{j+k} - t}{t_{j+k} - t_{j+1}} N_{j+1}^{k-1}(t), t \in \mathbb{R},$$

is called normalized B-spline basis function of order k (degree $k - 1$). The real numbers $t_j \leq t_{j+1}$ are called knot values or simply knots and $\frac{0}{0} \doteq 0$ by definition.

Definition 8 (B-spline curves) *The curve*

$$\mathbf{c}^k(t) = \sum_{j=0}^n \mathbf{d}_j N_j^k(t), t \in [t_{k-1}, t_{n+1}) \quad (10)$$

is called B-spline curve of order $1 \leq k \leq n + 1$ (degree $k - 1$), where $N_j^k(t)$ is the j th normalized B-spline basis function for the evaluation of which the knots

$$t_0 \leq t_1 \leq \dots \leq t_{n+k}$$

are necessary. The points $\mathbf{d}_j \in \mathbb{R}^m$ ($j = 0, 1, \dots, n; m \geq 2$) are called control points (de Boor-points), while the polygon

$$\mathbf{D} = [\mathbf{d}_j]_{j=0}^n \in \mathcal{M}_{1,n+1}(\mathbb{R}^m)$$

is called control polygon (de Boor-polygon). The arcs of this B-spline curve are called spans and the i th ($i = k - 1, k, \dots, n$) span can be expressed as

$$\mathbf{c}_i^k(t) = \sum_{j=i-k+1}^i \mathbf{d}_j N_j^k(t), t \in [t_i, t_{i+1}). \quad (11)$$

The normalized B-spline basis functions $N_j^k(t)$ are positive. Moreover, $N_j^k(t) = 0$ if $t \notin [t_j, t_{j+k})$ and the basis functions form a partition of the unity, i.e., $\sum_j N_j^k(t) \equiv 1, \forall t \in \mathbb{R}$. Thus, any point of the curve (10) is located in the convex hull of k consecutive control points. Moreover, the consecutive spans $\mathbf{c}_{i-1}^k(t)$ and $\mathbf{c}_i^k(t)$ ($i = k, \dots, n$) are C^{k-2} -continuous at the knot value $t = t_i$ provided that $k \geq 2$ and the multiplicity of t_i is one.

The following subsections describe the genetic representations of individuals and search operators involved in this genetic algorithm which proved to be useful for the optimization problem (9).

4.1 Genetic representation

Initial *individuals* of the *population* involved in the search process are randomly generated B-spline curves. Each individual is determined by an order k , a control polygon

$$\mathbf{Q} = [\mathbf{d}_0, \mathbf{d}_1, \dots, \mathbf{d}_n] = [\mathbf{d}_j]_{j=0}^n \in \mathcal{M}_{1,n+1}(\mathbb{R}^m)$$

and a knot vector

$$T = \{t_0, t_1, \dots, t_{n+k}\}.$$

Thus, an individual can be denoted as the triplet (k, \mathbf{Q}, T) .

Knot values

$$\begin{aligned} t_0 = t_1 = \dots = t_{k-1} = 0 < t_k < t_{k+1} < \dots < t_n \\ < t_{n+1} = t_{n+2} = \dots = t_{n+k} = 1 \end{aligned}$$

and control points

$$\mathbf{d}_j = \begin{cases} \mathbf{p}, & j = 0, \\ \mathbf{d}(j, \mathbf{p}, \mathbf{q}, \rho_j), & j \in \{1, 2, \dots, n-1\}, \\ \mathbf{q}, & j = n \end{cases}$$

ensure the required endpoint interpolation property, where the interior control points $\mathbf{d}_j = \mathbf{d}(j, \mathbf{p}, \mathbf{q}, \rho_j)$ ($j = 1, 2, \dots, n-1$) are selected randomly within the $m-1$ dimensional Euclidean balls

$$B^{m-1}(\mathbf{o}_j, \rho_j) = \{\mathbf{x} \in \mathbb{R}^{m-1} : |\mathbf{x} - \mathbf{o}_j| < \rho_j\}, \quad j = 1, 2, \dots, n-1$$

which are the subsets of the hyperplanes that are perpendicular (in Euclidean sense) to the direction determined by the points \mathbf{p}, \mathbf{q} and the origin of which are determined as

$$\mathbf{o}_j = \left(1 - \frac{j}{n}\right) \mathbf{p} + \frac{j}{n} \mathbf{q}, \quad j = 1, 2, \dots, n-1.$$

Figure 4 illustrates these balls in 2- and 3-dimensional cases. Naturally, one must also ensure that the image of all randomly generated individuals (B-spline curves) are the part of the Finsler manifold (M, F) . The radius ρ_j ($j = 1, 2, \dots, n-1$) of the Euclidean ball $B^{m-1}(\mathbf{o}_j, \rho_j)$ is a user-defined and manifold dependent fixed number.

Knot locations also effect the shape of the B-spline curve, but in this case the interior knot values t_k, t_{k+1}, \dots, t_n will be uniformly chosen, i.e.,

$$t_j = \frac{j-k+1}{n-k+2}, \quad j = k, k+1, \dots, n.$$

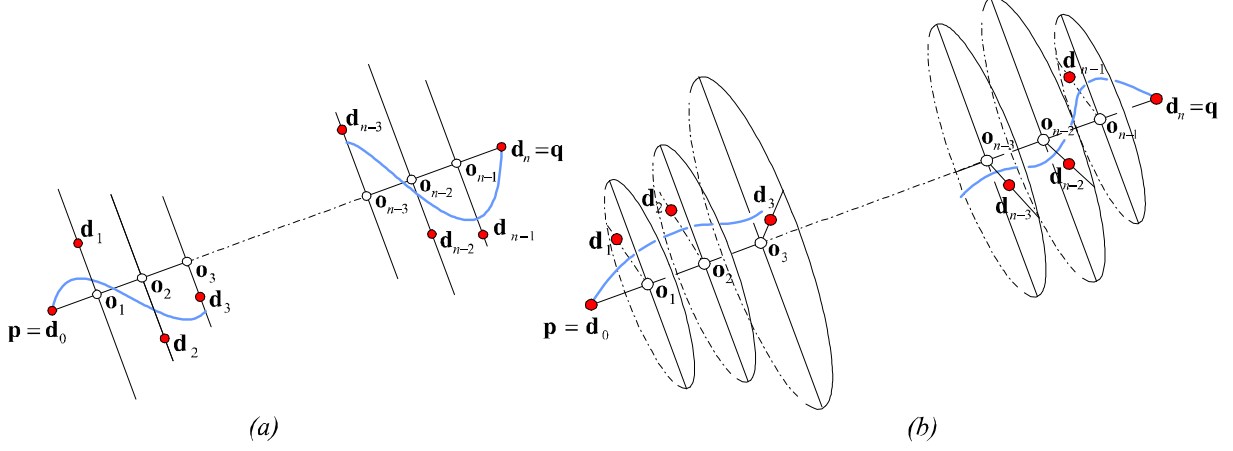


Figure 4: Generating an initial individual: (a) and (b) illustrate the domains of the possible interior control points in 2- and 3-dimensional cases.

The number $n + 1$ of control points, the order k and the knot vector T of individuals are the same for all individuals and they do not change during the evolutionary process. Thus, control points within a control polygon are the *genes* of an individual.

A search process starts with a randomly generated initial *population*

$$P_0 = \{(k, \mathbf{Q}_1, T), (k, \mathbf{Q}_2, T), \dots, (k, \mathbf{Q}_N, T)\}$$

consisting of the individuals

$$(k, \mathbf{Q}_l, T) = (k, [\mathbf{d}_0^l, \mathbf{d}_1^l, \dots, \mathbf{d}_n^l], T), \quad l = 1, 2, \dots, N.$$

As the population evolves from the current generation P_g ($g \geq 0$) to the next generation P_{g+1} , the number N of individuals remains the same.

The *fitness value* of the individual (k, \mathbf{Q}_l, T) , $l = 1, 2, \dots, N$ is given by

$$\text{eval}[(k, \mathbf{Q}_l, T)] = -L_F(\mathbf{c}_l^k),$$

where

$$\mathbf{c}_l^k(t) = \sum_{j=0}^n \mathbf{d}_j^l N_j^k(t), \quad t \in [t_{k-1}, t_{n+1})$$

is a B-spline curve of order k that consist of the spans

$$\mathbf{c}_{l,i}^k(t) = \sum_{j=i-k+1}^i \mathbf{d}_j^l N_j^k(t), \quad t \in [t_i, t_{i+1}), \quad i = k-1, k, \dots, n,$$

i.e.,

$$\begin{aligned} \text{eval}[(k, \mathbf{Q}_l, T)] &= -L_F(\mathbf{c}_l^k) \\ &= -\sum_{i=k-1}^n L_F(\mathbf{c}_{l,i}^k) \\ &= -\sum_{i=k-1}^n \int_{t_i}^{t_{i+1}} F(\mathbf{c}_{l,i}^k(t), \dot{\mathbf{c}}_{l,i}^k(t)) dt. \end{aligned}$$

Individual (k, \mathbf{Q}_i, T) is considered *better* than individual (k, \mathbf{Q}_j, T) if

$$\text{eval}[(k, \mathbf{Q}_i, T)] > \text{eval}[(k, \mathbf{Q}_j, T)].$$

4.2 Search operators

In the sequel we give a short description of search operators involved in the proposed genetic algorithm.

4.2.1 Selection

Tournament selection is used for determining the individuals of the next generations. This selection mechanism runs a "tournament" among a few individuals chosen randomly from the current population and selects the "winner" (i.e., the one with the best fitness) for recombination (or crossover). A number N of tournaments will result in N winner individuals that will be inserted into a mating pool. Individuals within this mating pool are used to generate new offsprings.

4.2.2 Recombination

Let us consider a randomly coupled pair $((k, \mathbf{Q}_i, T), (k, \mathbf{Q}_j, T))$ ($i \neq j, i, j \in \{1, 2, \dots, N\}$) of individuals (called *parents*) from the mating pool. In order to control this process, a recombination probability $p_{rec} \in [0, 1]$ can be used that determines how often will crossover be performed. Recombination is based on the following mating process: first we randomly select an interior point of the parents' control polygons, then the complementary parts of the parents' control points in arrays determined by the selected point are used to form the genes of the offsprings in the following way:

$$\left. \begin{array}{l} \mathbf{Q}_i = [\mathbf{d}_0^i, \mathbf{d}_1^i, \dots, \mathbf{d}_{l-1}^i, |\mathbf{d}_l^i, \dots, \mathbf{d}_n^i] \\ \mathbf{Q}_j = [\mathbf{d}_0^j, \mathbf{d}_1^j, \dots, \mathbf{d}_{l-1}^j, |\mathbf{d}_l^j, \dots, \mathbf{d}_n^j] \end{array} \right\} \implies \left\{ \begin{array}{l} \mathbf{Q}'_i = [\mathbf{d}_0^i, \mathbf{d}_1^i, \dots, \mathbf{d}_{l-1}^i, |\mathbf{d}_l^j, \dots, \mathbf{d}_n^j] \\ \mathbf{Q}'_j = [\mathbf{d}_0^j, \mathbf{d}_1^j, \dots, \mathbf{d}_{l-1}^j, |\mathbf{d}_l^i, \dots, \mathbf{d}_n^i] \end{array} \right.$$

The resulted offsprings (called *children*), the parents and the worst individual of the current population compete for survival. Naturally, one can also define multiple point recombination operators.

4.2.3 Mutation

Mutation is an operator used to maintain genetic diversity from the current generation of a population to the next one. This process can also be controlled by a mutation probability $p_{mut} \in (0, 1]$ that determines which genes will be altered.

Consider the randomly selected individual

$$(k, \mathbf{Q}_l, T) = (k, [\mathbf{d}_0^l, \mathbf{d}_1^l, \dots, \mathbf{d}_n^l], T), \quad l \in \{1, 2, \dots, N\},$$

and let us suppose that control point $\mathbf{d}_j^l, j \in \{1, \dots, n-1\}$ has also been randomly selected for mutation. The selected control point lies in a hyperplane that is perpendicular (in Euclidean sense) to the direction of $\mathbf{q} - \mathbf{p}$. Consider the $m-1$ dimensional Euclidean ball

$$B^{m-1}(\mathbf{d}_j^l, \rho_{mut}) = \{\mathbf{x} \in \mathbb{R}^{m-1} : |\mathbf{x} - \mathbf{d}_j^l| < \rho_{mut}\},$$

in the aforementioned hyperplane, where $\rho_{mut} > 0$ is called *maximal mutational radius*, which is either a fixed or a generation-based descending number that specifies the maximum deviation of the point being under mutation. A possible mutation is done by perturbing control point \mathbf{d}_j^l such that its new position will be somewhere in the Euclidean ball $B^{m-1}(\mathbf{d}_j^l, \rho_{mut})$. Naturally, we need also to ensure that after the perturbation process the image of the new individual (B-spline curve) will fall still in the manifold (M, F) .

Once again, the resulting offspring, the parent and the worst individual of the current population compete for survival.

Figure 5 depicts the evolution of four isolated populations the best individuals of which converge to the sides and the median of a geodesic triangle in the Finslerian-Poincaré disc.

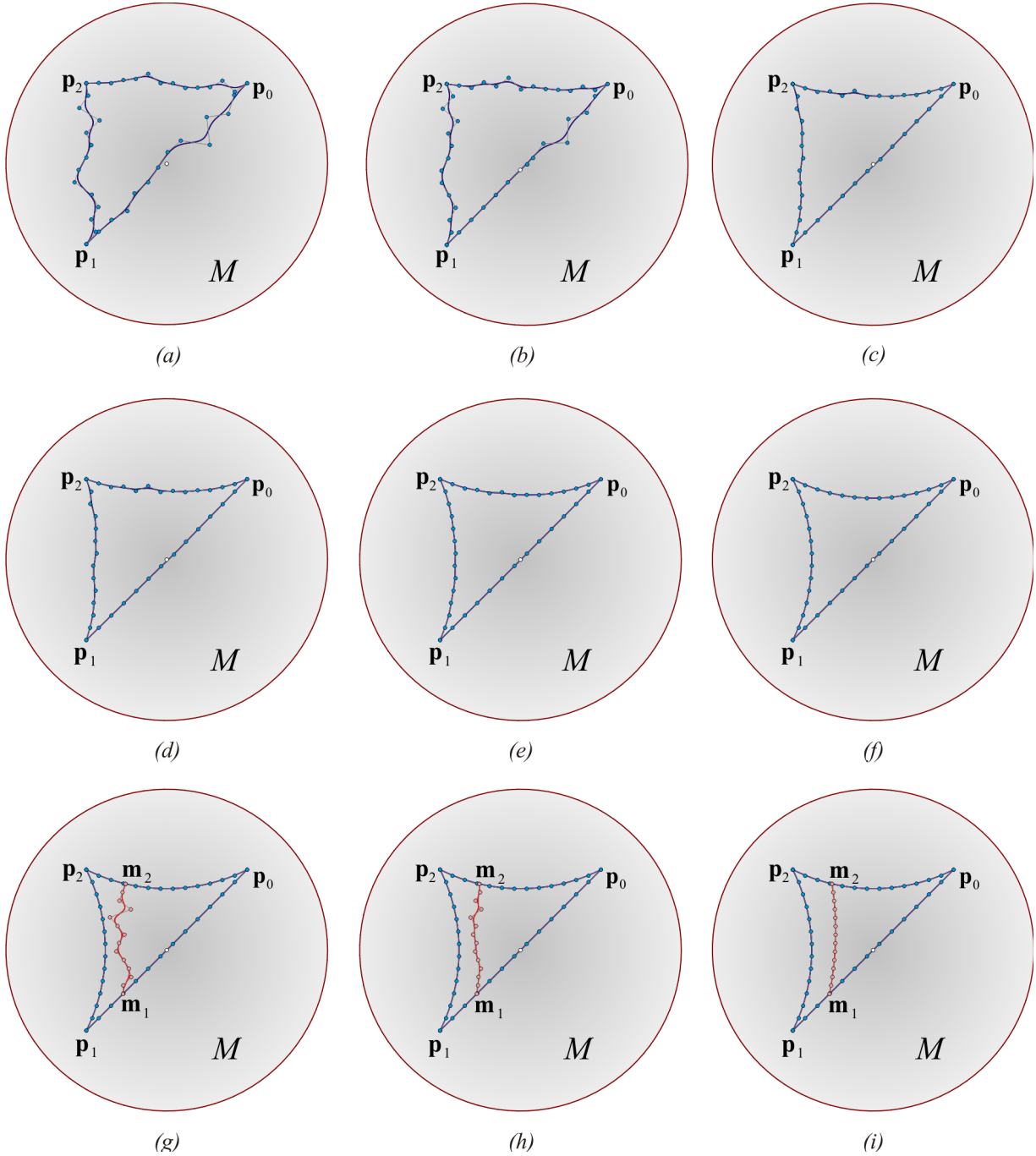


Figure 5: Evolution of the geodesic triangle in the Finslerian-Poincaré disc (see Figure 2). Three isolated populations of B-spline curves evolve to the expectant sides: one can follow up from (a) to (f) the evolving shape of the best individuals in consecutive generations of the corresponding populations. In case of the evolved geodesic triangle a new isolated population evolves the median (cf. images from (g) to (i)).

5 Final remarks

1. During the preparation of the manuscript we tested much more Finsler metrics (Berwald and non-Berwald structures) than those presented in the previous sections. We emphasize that the proposed numerical method can be efficiently applied to check various (local) metric relations on Finsler manifolds not only for Busemann-type inequalities.
2. The time and memory complexities of the proposed genetic algorithm heavily depend on the number of individuals of each isolated populations that evolve towards different geodesic lines. The evaluation of the points and derivatives of the B-spline curves depends on the order of the curves and on the size of the control polygons. The used quadrature formula, that approximates the theoretical value of the integral length needed for fitness evaluation, also effects the runtime of the algorithm. However, in case of the provided examples all geodesic triangles and their medians were obtained in reasonable time (i.e., approximately 40-120 seconds on a computer having an Intel Core 2 Duo 2.8 GHz processor).

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