

Resolvability of topological spaces

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1 Introduction

It is well known that there are $c = 2^\omega$ many disjoint dense subset of the real line.

On the other hand, it is easy to see that if, using Zorn's lemma, we take a maximal 0-dimensional and crowded refinement of the usual topology of the rational numbers then the resulting (countable) space has no two disjoint dense subset.

Numerous such questions may be asked about the resolvability of topological spaces into disjoint dense subsets. Such questions were first studied by E. Hewitt, [17], in 1943.

In the second section we introduce a ZFC method that enables us to build spaces (in fact special dense subspaces of certain Cantor cubes) in which we have "full control" over all dense subsets. Using this method we are able to construct, in ZFC, for each uncountable regular cardinal λ a 0-dimensional T_2 , hence Tychonov, space which is μ -resolvable for all $\mu < \lambda$ but not λ -resolvable. This yields the final (negative) solution of the following celebrated problem of Ceder and Pearson that was raised in 1967: Are ω -resolvable spaces maximally resolvable? This method enabled us to solve several other open problems concerning resolvability as well.

In the third section, we study the resolvability properties of spaces that have only small discrete or closed discrete subsets. In a recent paper O. Pavlov proved the following two interesting resolvability results:

1. If a space X satisfies $\Delta(X) > ps(X)$ then X is maximally resolvable.
2. If a T_3 -space X satisfies $\Delta(X) > pe(X)$ then X is ω -resolvable.

Here $\text{ps}(X)$ ($\text{pe}(X)$) denotes the smallest successor cardinal such that X has no discrete (closed discrete) subset of that size and $\Delta(X)$ is the smallest cardinality of a non-empty open set in X .

We improve 1. by showing that $\Delta(X) > \text{ps}(X)$ can be relaxed to $\Delta(X) \geq \text{ps}(X)$. In particular, if X is a space of countable spread with $\Delta(X) > \omega$ then X is maximally resolvable.

The question if an analogous improvement of 2. is valid remains open but we present here a proof of 2. that is simpler than Pavlov's.

In the final section we study resolvability properties of a special class of spaces, namely the *monotonically normal* spaces. We note that both metric and linearly ordered spaces are monotonically normal. We show that every crowded monotonically normal (in short: MN) space is ω -resolvable and almost μ -resolvable, where $\mu = \min\{2^\omega, \omega_2\}$. On the other hand, if κ is a measurable cardinal then there is a MN space X with $\Delta(X) = \kappa$ such that no subspace of X is ω_1 -resolvable.

1.1 Basic definitions and notation

Given a cardinal $\kappa > 1$, a topological space X is called κ -*resolvable* iff it contains κ disjoint dense subsets. X is κ -*irresolvable* iff it is not κ -resolvable. X is *resolvable* iff it is 2-resolvable and *irresolvable* otherwise.

X is called ($< \kappa$)-*resolvable* iff for every $\mu < \kappa$ X is μ -resolvable.

If X is κ -resolvable and $G \subset X$ is any non-empty open set in X then clearly

$\kappa \leq |G|$. Hence if X is κ -resolvable then we have $\kappa \leq \Delta(X)$ where

$$\Delta(X) = \min\{|G| : G \text{ is a nonempty open set}\}.$$

This observation explains the following terminology of J.Ceder, [5]: a space X is called *maximally resolvable* iff it is $\Delta(X)$ -resolvable.

A space X is called *open hereditarily irresolvable (OHI)* iff every nonempty open subspace of X is irresolvable. It is well-known that every irresolvable space has a non-empty open subspace that is OHI. Clearly, X is OHI iff every dense subset of X contains a dense open subset, i. e. if $S \subset X$ dense in X implies that $\text{Int}(S)$ is dense, as well.

Next, a space X is called *hereditarily irresolvable(HI)* iff all subspaces of X are irresolvable. Since a space having an isolated point is trivially irresolvable, any space is HI iff all its *crowded* subspaces are irresolvable. (Following van Douwen, we call a space crowded if it has no isolated points.) Having this in mind, if P is any resolvability or irresolvability property of topological spaces then the space X is called *hereditarily P* iff all *crowded* subspaces of X have property P .

Following the terminology of [34], a topological space X is called *NODEC* if all nowhere dense subsets of X are closed, and hence closed discrete. All spaces obtained by our main theorem 2.13 will be NODEC.

A space is called *submaximal* (see [17]) iff all of its dense subsets are open. The following observation is easy to prove and will be used repeatedly later: a space is submaximal iff it is both OHI and NODEC.

A set $D \subset X$ is said to be κ -dense in X iff $|D \cap U| \geq \kappa$ for each nonempty

open set $U \subset X$. Thus D is dense iff it is 1-dense. Also, it is obvious that the existence of a κ -dense set in X implies $\Delta(X) \geq \kappa$.

We shall denote by $\mathcal{N}(X)$ the family of all nowhere dense subsets of a space X . Clearly, $\mathcal{N}(X)$ is an ideal of subsets of X and the notation $=^*$ or \subset^* will always be used to denote equality, resp. inclusion modulo this ideal.

Following the notation introduced in [6], we shall write

$$\text{nwd}(X) = \min\{|Y| : Y \in \mathcal{P}(X) \setminus \mathcal{N}(X)\} = \text{non} - (\mathcal{N}(X)),$$

i. e. $\text{nwd}(X)$ is the minimum cardinality of a somewhere dense subset of X .

Malychin was the first to suggest studying families of dense sets of a space X that are *almost disjoint* with respect to the ideal $\mathcal{N}(X)$ rather than disjoint, see [29]. He calls a space X *extraresolvable* if there are $\Delta(X)^+$ many dense sets in X such that any two of them have nowhere dense intersection. Here we generalize this concept by defining a space X to be *κ -almost-resolvable* if there are κ many dense sets in X such that any two of them have nowhere dense intersection. Note that, although κ -almost-resolvability of X is mainly of interest if $\kappa > \Delta(X)$, it does make sense for $\kappa \leq \Delta(X)$ as well. Clearly, κ -resolvable implies κ -almost-resolvable, moreover the converse holds if $\kappa = \omega$.

2 Separating various resolvability properties

2.1 \mathcal{D} -forced spaces

Definition 2.1. Let \mathcal{D} be a family of dense subsets of a space X . A subset $M \subset X$ is called a (\mathcal{D}, X) -*mosaic* iff there is a maximal disjoint family \mathcal{V} of open subsets of X and for each $V \in \mathcal{V}$ there is $D_V \in \mathcal{D}$ such that

$$M = \cup\{V \cap D_V : V \in \mathcal{V}\}.$$

A set M of the above form with \mathcal{V} disjoint, but not necessarily maximal disjoint, is called a *partial* (\mathcal{D}, X) -mosaic.

A set P of the form $P = D \cap U$, where $D \in \mathcal{D}$ and U is a nonempty open subset of X , is called a (\mathcal{D}, X) -*piece*. So, naturally, any (partial) (\mathcal{D}, X) -mosaic is composed of (\mathcal{D}, X) -pieces. Let

$$\mathfrak{M}(\mathcal{D}, X) = \{M : M \text{ is a } (\mathcal{D}, X)\text{-mosaic}\}$$

and

$$\mathcal{P}(\mathcal{D}, X) = \{P : P \text{ is a } (\mathcal{D}, X)\text{-piece}\}.$$

When the space X is clear from the context we will omit it from the notation: we will write \mathcal{D} -*mosaic* instead of (\mathcal{D}, X) -mosaic, and \mathcal{D} -*piece* instead of (\mathcal{D}, X) -piece, etc. The following statement is now obvious.

Fact 2.2. *Every (\mathcal{D}, X) -mosaic is dense in X and every (\mathcal{D}, X) -piece is somewhere dense in X .*

Thus we arrive at the following very simple but, as it turns out, very useful concept.

Definition 2.3. Let \mathcal{D} be a family of dense subsets of a topological space X . We say that the space X (or its topology) is \mathcal{D} -forced iff every dense subset S of X includes a \mathcal{D} -mosaic M , i. e. there is $M \in \mathfrak{M}(\mathcal{D}, X)$ with $M \subset S$.

It is easy to check that one can give the following alternative characterization of being \mathcal{D} -forced.

Fact 2.4. *The space X is \mathcal{D} -forced iff every somewhere dense subset of X includes a (\mathcal{D}, X) -piece.*

Since X is always dense in X , the simplest choice for \mathcal{D} is $\{X\}$.

Fact 2.5. *A subset $P \subset X$ is an $\{X\}$ -piece iff it is non-empty open; M is an $\{X\}$ -mosaic iff it is dense open in X . Consequently, X is $\{X\}$ -forced iff it is OHI.*

Let us now consider a few further, somewhat less obvious, properties of \mathcal{D} -forced spaces. The first result yields a useful characterization of nowhere dense subsets in such spaces. Note that a subset Y of any space X is nowhere dense iff $S \setminus Y$ is dense in X for all dense subsets S of X . Not surprisingly, in a \mathcal{D} -forced space it suffices to check this for members of \mathcal{D} .

Lemma 2.6. *Assume that X is \mathcal{D} -forced. Then*

$$\mathcal{N}(X) = \{Y \subset X : D \setminus Y \text{ is dense in } X \text{ for each } D \in \mathcal{D}\}.$$

Proof. Assume that $Y \notin \mathcal{N}(X)$, i.e. Y is somewhere dense. Then, by fact 2.4, Y contains some \mathcal{D} -piece $U \cap D$, where $D \in \mathcal{D}$ and U is a nonempty open subset of X . Then $(D \setminus Y) \cap U = \emptyset$, i.e. $D \setminus Y$ is not dense. This proves that the right-hand side of the equality includes the left one. The converse inclusion is obvious. \square

The following result will be used to produce irreducible (even OHI) spaces. Of course, the superscript $*$ in its formulation designates equality and inclusion modulo the ideal $\mathcal{N}(X)$ of nowhere dense sets.

Lemma 2.7. *Let X be \mathcal{D} -forced and $S \subset X$ be dense such that*

(\dagger) *for each $D \in \mathcal{D}$ we have $S \cap D =^* \emptyset$ or $S \subset^* D$.*

Then S , as a subspace of X , is OHI.

Proof. Let $T \subset S$ be dense in S , then T is also dense in X , hence it must contain a \mathcal{D} -mosaic, say $M = \bigcup \{V \cap D_V : V \in \mathcal{V}\}$. But then we have $S \subset^* D_V$ for each $V \in \mathcal{V}$ by (\dagger). Consequently,

$$T \cap V \subset S \cap V \subset^* V \cap D_V \subset T \cap V$$

and so $T \cap V =^* S \cap V$ holds for all $V \in \mathcal{V}$. This clearly implies that $T =^* S$. In other words, we have shown that every dense subset T of S has nowhere dense complement in S , i. e. the subspace S of X is OHI. \square

The following lemma will enable us to conclude that certain \mathcal{D} -forced spaces are not κ -(extra)resolvable for appropriate cardinals κ .

Lemma 2.8. *Assume that X is a topological space and \mathcal{D} is a family of dense subsets of X . Assume, moreover, that $\mu \geq \widehat{c}(X)$ (i.e. X does not contain μ many pairwise disjoint open subsets) and*

(*) *for each $\mathcal{E} \in [\mathcal{D}]^\mu$ there is $\mathcal{F} \in [\mathcal{E}]^{\widehat{c}(X)}$ such that*

$$D_0 \cap D_1 \text{ is dense in } X \text{ whenever } \{D_0, D_1\} \in [\mathcal{F}]^2.$$

Then for any family of \mathcal{D} -pieces $\{P_i : i < \mu\} \subset \mathcal{P}(\mathcal{D})$ there is $\{i, j\} \in [\mu]^2$ such that $P_i \cap P_j$ is somewhere dense in X .

In particular, if X is \mathcal{D} -forced and $|\mathcal{D}|^+ \geq \widehat{c}(X)$ then X is not $|\mathcal{D}|^+$ -almost-resolvable (hence not $|\mathcal{D}|^+$ -resolvable, either).

Proof. Assume that $P_i = U_i \cap D_i$, where $D_i \in \mathcal{D}$ and U_i is a nonempty open subset of X for all $i \in \mu$. By (*) there is $I \in [\mu]^{\widehat{c}(X)}$ such that $D_i \cap D_j$ is dense for each $\{i, j\} \in [I]^2$. By the definition of $\widehat{c}(X)$, there is $\{i, j\} \in [I]^2$ such that $U = U_i \cap U_j$ is non-empty. But then $U \cap D_i \cap D_j \subset P_i \cap P_j$, hence $P_i \cap P_j$ is dense in the nonempty open set U .

The last statement now follows because \mathcal{D} trivially satisfies condition (*) with $\mu = |\mathcal{D}|^+$ and, as X is \mathcal{D} -forced, every dense subset of X includes a \mathcal{D} -piece (even a \mathcal{D} -mosaic). □

The following fact is obvious.

Fact 2.9. *Let \mathcal{D} be a family of dense sets in X and*

$$M = \bigcup \{V \cap D_V : V \in \mathcal{V}\}$$

be a partial \mathcal{D} -mosaic. If all the dense sets D_V are μ -(extra)resolvable for $V \in \mathcal{V}$ then so is M .

We finish this section with a result that, together with fact 2.9, will be used to establish hereditary (extra)resolvability properties of several examples constructed later.

Lemma 2.10. *Let X be a \mathcal{D} -forced space in which every crowded subspace is somewhere dense. (This holds e. g. if X is NODEC.) Then for every crowded $S \subset X$ there is a partial \mathcal{D} -mosaic $M \subset S$ that is dense in S . So if, in addition, all $D \in \mathcal{D}$ are μ -resolvable (resp. μ -almost-resolvable) then X is hereditarily μ -resolvable (resp. μ -almost-resolvable).*

Proof. Let \mathcal{V} be a maximal disjoint family of open sets V such that there is an element $D_V \in \mathcal{D}$ with $V \cap D_V \subset S$ and consider the partial \mathcal{D} -mosaic

$$M = \bigcup \{V \cap D_V : V \in \mathcal{V}\}.$$

Then $M \subset S$ is dense in S , since otherwise, in view of the maximality of \mathcal{V} , the set $S \setminus \overline{M} \neq \emptyset$ would be crowded and could not include any \mathcal{D} -piece. The last sentence now immediately follows using fact 2.9. \square

2.2 The Main Theorem

We have introduced the concept of \mathcal{D} -forced spaces but one question that immediately will be raised is if there are any beyond the obvious choice of $\mathcal{D} = \{X\}$?

The aim of this section is to prove theorem 2.13 that provides us with a large supply of such spaces. All these spaces will be dense subspaces of Cantor cubes, i. e. powers of the discrete two-point space $D(2)$. As is well-known, there is a natural one-to-one correspondence between dense subspaces of size κ of the Cantor cube $D(2)^\lambda$ and independent families of 2-partitions of κ indexed by λ . (A partition of a set S is called a μ -partition if it partitions S into μ many pieces.) For technical reasons, we shall produce our spaces by using partitions rather than Cantor cubes.

We start with fixing some notation and terminology.

Let $\vec{\lambda} = \langle \lambda_\zeta : \zeta < \mu \rangle$ be a sequence of cardinals. We set

$$\begin{aligned} \mathbb{FIN}(\vec{\lambda}) &= \{ \varepsilon : \varepsilon \text{ is a finite function with } \text{dom } \varepsilon \in [\mu]^{<\omega} \text{ and} \\ &\quad \varepsilon(\zeta) \in \lambda_\zeta \text{ for all } \zeta \in \text{dom } \varepsilon \}. \end{aligned}$$

Note that if $\lambda_\zeta = \lambda$ for all $\zeta < \mu$ then

$$\mathbb{FIN}(\vec{\lambda}) = \text{Fn}(\mu, \lambda).$$

Let S be a set, $\vec{\lambda} = \langle \lambda_\zeta : \zeta < \mu \rangle$ be a sequence of cardinals, and $\mathbb{B} = \{ \langle B_\zeta^i : i < \lambda_\zeta \rangle : \zeta < \mu \}$ be a family of partitions of S . Given a cardinal κ we say that \mathbb{B} is κ -independent iff

$$\mathbb{B}[\varepsilon] \stackrel{\text{def}}{=} \bigcap \{ B_\zeta^{\varepsilon(\zeta)} : \zeta \in \text{dom } \varepsilon \}$$

has cardinality at least κ for each $\varepsilon \in \mathbb{FIN}(\vec{\lambda})$. \mathbb{B} is *independent* iff it is 1-independent, i.e. $\mathbb{B}[\varepsilon] \neq \emptyset$ for each $\varepsilon \in \mathbb{FIN}(\vec{\lambda})$. \mathbb{B} is *separating* iff for each $\{\alpha, \beta\} \in [S]^2$ there are $\zeta < \mu$ and $\{\rho, \nu\} \in [\lambda_\zeta]^2$ such that $\alpha \in B_\zeta^\rho$ and $\beta \in B_\zeta^\nu$.

We shall denote by $\tau_{\mathbb{B}}$ the (obviously zero-dimensional) topology on S generated by the subbase $\{B_{\zeta}^i : \zeta < \mu, i < \lambda_{\zeta}\}$, moreover we set $X_{\mathbb{B}} = \langle S, \tau_{\mathbb{B}} \rangle$. Clearly, the family $\{\mathbb{B}[\varepsilon] : \varepsilon \in \mathbb{FIN}(\vec{\lambda})\}$ is a base for the space $X_{\mathbb{B}}$. Note that $X_{\mathbb{B}}$ is Hausdorff iff \mathbb{B} is separating.

The following statement is very easy to prove and is well-known. It can certainly be viewed as part of the folklore.

Observation 2.11. *Let κ and λ be infinite cardinals. Then, up to homeomorphisms, there is a natural one-to-one correspondence between dense subspaces X of $D(2)^{\lambda}$ of size κ and spaces of the form $X_{\mathbb{B}} = \langle \kappa, \tau_{\mathbb{B}} \rangle$, where $\mathbb{B} = \{\langle B_{\xi}^0, B_{\xi}^1 \rangle : \xi < \lambda\}$ is a separating and independent family of 2-partitions of κ . Moreover, X is μ -dense in $D(2)^{\lambda}$ iff \mathbb{B} is μ -independent.*

The spaces obtained from our main theorem 2.13 will all be of the above form, with $\lambda = 2^{\kappa}$. The following fact will be instrumental in finding appropriate families of dense sets \mathcal{D} to be used to produce \mathcal{D} -forced spaces.

Fact 2.12. *For each infinite cardinal κ , there is a family*

$$\mathbb{B} = \{\langle B_{\xi}^i : i < \kappa \rangle : \xi < 2^{\kappa}\}$$

of 2^{κ} many κ -partitions of κ that is κ -independent.

Indeed, this fact is just a reformulation of the statement that the space $D(\kappa)^{2^{\kappa}}$, the 2^{κ} th power of the discrete space on κ , contains a κ -dense subset of size κ . This, in turn, follows immediately from the Hewitt-Marczewski-Pondiczery theorem, see e. g. [13, theorem 2.3.15].

Main theorem 2.13. *Assume that κ is an infinite cardinal and we are given $\mathbb{B} = \{\langle B_\xi^0, B_\xi^1 \rangle : \xi < 2^\kappa\}$, a κ -independent family of 2-partitions of κ , moreover a non-empty family \mathcal{D} of κ -dense subsets of the space $X_{\mathbb{B}}$. Then there is another, always separating, κ -independent family $\mathbb{C} = \{\langle C_\xi^0, C_\xi^1 \rangle : \xi < 2^\kappa\}$ of 2-partitions of κ that satisfies the following five conditions:*

- (1) *every $D \in \mathcal{D}$ is also κ -dense in $X_{\mathbb{C}}$ (and so $\Delta(X_{\mathbb{C}}) = \kappa$),*
- (2) *$X_{\mathbb{C}}$ is \mathcal{D} -forced,*
- (3) *$\text{nwd}(X_{\mathbb{C}}) = \kappa$, i.e. $[\kappa]^{<\kappa} \subset \mathcal{N}(X_{\mathbb{C}})$,*
- (4) *$X_{\mathbb{C}}$ is NODEC.*

Moreover, if $J \subset 2^\kappa$ is given with $|2^\kappa \setminus J| = 2^\kappa$ then we can assume that

- (5) *$\mathbb{C} \upharpoonright J = \mathbb{B} \upharpoonright J$.*

Proof. Assume that J is given and let $I = 2^\kappa \setminus J$. We partition I into two disjoint pieces, $I = I_0 \cup I'$, such that $|I_0| = \kappa^{<\kappa}$ and $|I'| = 2^\kappa$. Next we partition I_0 into pairwise disjoint countable sets $J_{A,\alpha} \in [I_0]^\omega$ for all $A \in [\kappa]^{<\kappa}$ and $\alpha \in \kappa \setminus A$. If $\xi \in J_{A,\alpha}$ (for some $A \in [\kappa]^{<\kappa}$ and $\alpha \in \kappa \setminus A$) then we let

$$C_\xi^0 = (B_\xi^0 \cup A) \setminus \{\alpha\},$$

and

$$C_\xi^1 = (B_\xi^1 \setminus A) \cup \{\alpha\}.$$

Next, let us fix any enumeration $\{F_\nu : \nu < 2^\kappa\}$ of $[\kappa]^\kappa$ and then by transfinite recursion on $\nu < 2^\kappa$ define

- sets $K_\nu \subset I'$ with $K_\nu = \emptyset$ or $|K_\nu| = \kappa$,
- partitions $\langle C_\sigma^0, C_\sigma^1 \rangle$ of κ for all $\sigma \in K_\nu$,
- finite functions $\eta_\nu \in \text{Fn}(2^\kappa, 2)$,

such that the inductive hypothesis

$$(\phi_\nu) \quad \forall \varepsilon \in \text{Fn}(2^\kappa, 2) \quad \forall D \in \mathcal{D} \quad |D \cap \mathbb{B}_\nu[\varepsilon]| = \kappa$$

holds, where

$$\mathbb{B}_\nu = \left\{ \langle C_\sigma^0, C_\sigma^1 \rangle : \sigma \in I_\nu \right\} \cup \left\{ \langle B_\sigma^0, B_\sigma^1 \rangle : \sigma \in 2^\kappa \setminus I_\nu \right\}$$

with

$$I_\nu = I_0 \cup \bigcup_{\zeta < \nu} K_\zeta.$$

Note that (ϕ_ν) simply says that every set $D \in \mathcal{D}$ is κ -dense in the space $X_{\mathbb{B}_\nu}$.

We shall then conclude that $\mathbb{C} = \mathbb{B}_{2^\kappa}$ is as required.

Let us observe first that (ϕ_0) holds because, by assumption, we have $|\mathbb{B}[\varepsilon] \cap D| = \kappa$ for all $D \in \mathcal{D}$ and $\varepsilon \in \text{Fn}(2^\kappa, 2)$, moreover

$$|\mathbb{B}[\varepsilon] \triangle \mathbb{B}_0[\varepsilon]| < \kappa.$$

Clearly, if ν is a limit ordinal and (ϕ_ζ) holds for each $\zeta < \nu$ then (ϕ_ν) also holds. So the induction hypothesis is preserved in limit steps.

Now consider the successor steps. Assume that (ϕ_ν) holds. We distinguish two cases:

Case 1. F_ν contains a $(\mathcal{D}, X_{\mathbb{B}_\nu})$ -piece, i.e. $F_\nu \supset D \cap \mathbb{B}_\nu[\eta_\nu]$ for some $\eta_\nu \in \text{Fn}(2^\kappa, 2)$ and $D \in \mathcal{D}$.

This defines η_ν and then we set $K_\nu = \emptyset$. The construction from here on will not change the partitions whose indices occur in $\text{dom}(\eta_\nu)$, thus we shall have $\mathbb{B}_\nu[\eta_\nu] = \mathbb{B}_{2^\kappa}[\eta_\nu]$ and so at the end F_ν will include the $(\mathcal{D}, X_{\mathbb{B}_{2^\kappa}})$ -piece $D \cap \mathbb{B}_{2^\kappa}[\eta_\nu]$. Also, in this case we have $\mathbb{B}_\nu = \mathbb{B}_{\nu+1}$, hence $(\phi_{\nu+1})$ trivially remains valid.

Case 2. F_ν does not include a $(\mathcal{D}, X_{\mathbb{B}_\nu})$ -piece, i.e. $(D \cap \mathbb{B}_\nu[\varepsilon]) \setminus F_\nu \neq \emptyset$ for all $\varepsilon \in \text{Fn}(2^\kappa, 2)$ and $D \in \mathcal{D}$.

In this case we choose and fix any set

$$K_\nu \subset I' \setminus (\cup\{\text{dom } \eta_\zeta : \zeta < \nu\} \cup \cup\{K_\zeta : \zeta < \nu\})$$

of size κ and let $K_\nu = \{\gamma_{\nu,i} : i < \kappa\}$ be a 1-1 enumeration of K_ν . We also set $\eta_\nu = \emptyset$. We want to modify the partitions with indices in K_ν so as to make the set F_ν closed discrete in $X_{\mathbb{B}_{\nu+1}}$ and hence in $X_{\mathbb{B}_{2^\kappa}}$ as well. To do this, we set for all $i < \kappa$

$$C_{\gamma_{\nu,i}}^0 = (B_{\gamma_{\nu,i}}^0 \setminus F_\nu) \cup \{i\},$$

and

$$C_{\gamma_{\nu,i}}^1 = (B_{\gamma_{\nu,i}}^1 \cup F_\nu) \setminus \{i\}.$$

Then for each $i \in \kappa$ we have $i \in C_{\gamma_{\nu,i}}^0$ and

$$F_\nu \cap C_{\gamma_{\nu,i}}^0 \subset \{i\},$$

consequently F_ν is closed discrete in $X_{\mathbb{B}_{\nu+1}}$, hence F_ν will be closed discrete in $X_{\mathbb{B}_{2^\kappa}}$.

We still have to show that $(\phi_{\nu+1})$ holds in this case, too. Assume, indirectly, that for some $D \in \mathcal{D}$ and $\varepsilon \in \text{Fn}(2^\kappa, 2)$ we have

$$|D \cap \mathbb{B}_{\nu+1}[\varepsilon]| < \kappa.$$

Then we can clearly find $\xi \in I_0 \setminus \text{dom } \varepsilon$ with

$$(D \cap \mathbb{B}_{\nu+1}[\varepsilon]) \cup \text{dom}(\varepsilon) \subset C_\xi^0,$$

and so for $\varepsilon^* = \varepsilon \cup \{\langle \xi, 1 \rangle\}$ we even have

$$D \cap \mathbb{B}_{\nu+1}[\varepsilon^*] = \emptyset.$$

On the other hand, our choices clearly imply that

$$\mathbb{B}_{\nu+1}[\varepsilon^*] \supset \mathbb{B}_\nu[\varepsilon^*] \setminus F_\nu,$$

consequently

$$D \cap \mathbb{B}_{\nu+1}[\varepsilon^*] \supset (D \cap \mathbb{B}_\nu[\varepsilon^*]) \setminus F_\nu \neq \emptyset,$$

a contradiction. This shows that $(\phi_{\nu+1})$ is indeed valid, and the transfinite construction of $\mathbb{C} = \mathbb{B}_{2^\kappa}$ is thus completed. We show next that \mathbb{C} satisfies all the requirements of our main theorem.

\mathbb{C} is separating because e. g. for any $\xi \in J_{\{\alpha\}, \beta}$ the partition $\langle C_\xi^0, C_\xi^1 \rangle$ separates α and β .

That \mathbb{C} is κ -independent and that (1) holds (i. e. each $D \in \mathcal{D}$ is κ -dense in $X_{\mathbb{C}}$) both follow from (ϕ_{2^κ}) .

If $A \in [\kappa]^{<\kappa}$ and $\alpha \in \kappa \setminus A$, then for any $\xi \in J_{A,\alpha}$ we have $A \subset C_\xi^0$ and $\alpha \in C_\xi^1$, hence $\alpha \notin \overline{A}^{X_{\mathbb{C}}}$. Thus every member of $[\kappa]^{<\kappa}$ is closed and hence closed discrete in $X_{\mathbb{C}}$, and so (3) is satisfied.

Assume next that $F \in \mathcal{N}(X_{\mathbb{C}})$, we want to show that F is closed discrete. By (3) we may assume that $|F| = \kappa$ and so can find $\nu < 2^\kappa$ with $F = F_\nu$. Suppose that at step ν of the recursion we were in case 1; then we had $F \supset D \cap \mathbb{B}_\nu[\eta_\nu]$ for some $D \in \mathcal{D}$. But $\mathbb{B}_\nu[\eta_\nu] = \mathbb{B}_{2^\kappa}[\eta_\nu] = \mathbb{C}[\eta_\nu]$, so F would be dense in $\mathbb{C}[\eta_\nu]$. This contradiction shows that, at step ν , we must have been in case 2. However, in this case we know that $F = F_\nu$ was made to be closed discrete in $X_{\mathbb{B}_{\nu+1}}$ and consequently in $X_{\mathbb{C}}$ as well. So $X_{\mathbb{C}}$ is NODEC, i.e. (4) holds.

It remains to check that $X_{\mathbb{C}}$ is \mathcal{D} -forced, i. e. that (2) holds. By 2.4 it suffices to show that any somewhere dense subset E of $X_{\mathbb{C}}$ includes a $(\mathcal{D}, X_{\mathbb{C}})$ -piece. By (3) we must have $|E| = \kappa$ and hence we can pick $\nu < 2^\kappa$ such that $F_\nu = E$. Then at step ν of the recursion we could not be in case 2, since otherwise $F_\nu = E$ would have been made closed discrete in $X_{\mathbb{B}_{\nu+1}}$ and so in $X_{\mathbb{C}}$ as well. Hence at step ν of the recursion we were in case 1, consequently $\eta_\nu \in \text{Fn}(2^\kappa, 2)$ and $D \in \mathcal{D}$ could be found such that $E = F_\nu \supset D \cap \mathbb{B}_\nu[\eta_\nu]$. However, by the construction, we have $\mathbb{C}[\eta_\nu] = \mathbb{B}_\nu[\eta_\nu]$, and therefore E actually includes the $(\mathcal{D}, X_{\mathbb{C}})$ -piece $D \cap \mathbb{C}[\eta_\nu]$.

Finally, (5) trivially holds by the construction.

□

2.3 Applications to resolvability

In this and the following part of the section we shall present a large number of consequences of our main theorem 2.13. The key to most of these will be given by a judicious choice of a family \mathcal{D} of κ -dense sets in a space $X_{\mathbb{B}}$, where $\mathbb{B} = \{\langle B_{\xi}^0, B_{\xi}^1 \rangle : \xi < 2^{\kappa}\}$ is a κ -independent family of 2-partitions of some cardinal κ . In our first application, however, this choice is trivial, that is we have $\mathcal{D} = \{\kappa\}$.

In [2], the following results were proven:

- (1) $D(2)^{\mathfrak{c}}$ does not have a dense countable maximal subspace,
- (2) $D(2)^{\mathfrak{c}}$ has a dense countable irresolvable subspace,
- (3) it is consistent that $D(2)^{\mathfrak{c}}$ has a dense countable submaximal subspace,

and then the following natural problem was raised ([2, Question 4.4]): *Is it provable in ZFC that the Cantor cube $D(2)^{\mathfrak{c}}$ or the Tychonov cube $[0, 1]^{\mathfrak{c}}$ has a dense countable submaximal subspace?* Our next result gives an affirmative answer to this problem.

Theorem 2.14. *For each infinite cardinal κ the Cantor cube $D(2)^{2^{\kappa}}$ contains a dense submaximal subspace X with $|X| = \Delta(X) = \kappa$.*

Proof. Let us start by fixing any κ -independent family of 2-partitions $\mathbb{B} = \{\langle B_{\xi}^0, B_{\xi}^1 \rangle : \xi < 2^{\kappa}\}$ of κ , and let $\mathcal{D} = \{\kappa\}$. Applying theorem 2.13 with \mathbb{B} and \mathcal{D} we obtain a family of 2-partitions \mathbb{C} of κ that satisfies 2.13 (1)–(4). The space $X_{\mathbb{C}}$ is as required. Indeed, $\Delta(X_{\mathbb{C}}) = \kappa$ because of 2.13(1), $X_{\mathbb{C}}$ is NODEC by 2.13(4), while

it is OHI by lemma 2.7. But then it is submaximal. Finally, by observation 2.11, $X_{\mathbb{C}}$ embeds into $D(2)^{2^{\kappa}}$ as a dense subspace. $\square_{2.14}$

That theorem 2.14 fully answers [2, Question 4.4] follows from the following fact 2.15 that may be already known, although we have not found it in the literature.

Fact 2.15. *Any countable dense subspace of $D(2)^{\mathfrak{c}}$ is homeomorphic to a dense subspace of $[0, 1]^{\mathfrak{c}}$.*

This fact, in turn, immediately follows from the next proposition. In it, as usual, we denote by \mathbb{P} the space of the irrationals.

Proposition 2.16. *Assume that κ is an infinite cardinal, $S \subset D(2)^{\kappa}$ is dense, moreover there is a partition $\{I_{\nu} : \nu < \kappa\}$ of κ into countably infinite sets such that for each $\nu < \kappa$ the set $2^{I_{\nu}} \setminus (S \upharpoonright I_{\nu})$ is dense (in other words: $S \upharpoonright I_{\nu}$ is co-dense) in $2^{I_{\nu}}$. (The last condition is trivially satisfied if the cardinality of S is less than continuum.) Then S is homeomorphic to a dense subspace of the irrational cube \mathbb{P}^{κ} and hence of the Tychonov cube $[0, 1]^{\kappa}$.*

Proof. For each $\nu < \kappa$ we may select a countable dense subset of $D_{\nu} \subset 2^{I_{\nu}} \setminus (S \upharpoonright I_{\nu})$. The space $2^{I_{\nu}} \setminus D_{\nu}$ is known to be homeomorphic to \mathbb{P} for all $\nu < \kappa$. Also, for each $\nu < \kappa$ we have $S \upharpoonright I_{\nu} \subset 2^{I_{\nu}} \setminus D_{\nu}$ and therefore S is naturally homeomorphic to a dense subspace of the product space

$$\prod \{2^{I_{\nu}} \setminus D_{\nu} : \nu < \kappa\}.$$

This product, however, is homeomorphic to the cube \mathbb{P}^{κ} . \square

Let us remark that, as far as we know, the first ZFC example of a countable regular, hence 0-dimensional, submaximal space was constructed by E. van Douwen in [34], by using an approach that is very different from and much more involved than ours. Also, it is not clear if his example embeds *densely* into the Cantor or Tychonov cube of weight \mathfrak{c} .

After proving in [1, Corollary 8.5] that every separable submaximal topological group is countable, Arhangel'skii and Collins raised the following question [1, Problem 8.6]: *Is there a crowded uncountable separable Hausdorff (or even Tychonov) submaximal space?* As it turns out, starting from any zero-dimensional countable submaximal space (e. g. the one obtained from the previous theorem or van Douwen's example from [34]) an affirmative answer can be given to this question, at least in the T_2 case. The regular or Tychonov cases of the problem, however, remain open.

Theorem 2.17. *There is a crowded, separable, submaximal T_2 space Y of cardinality \mathfrak{c} .*

Proof. Let τ be any fixed crowded, submaximal, 0-dimensional, and T_2 topology on ω . Since τ is not compact we can easily find $\{U_\sigma : \sigma \in 2^{<\omega}\}$, an infinite partition of ω into nonempty τ -clopen sets indexed by all finite 0-1 sequences σ .

The underlying set of Y will be $\omega \cup {}^\omega 2$ and we let $X = \langle \omega, \tau \rangle$ be an open subspace of Y . Next, a basic neighbourhood of a point $f \in {}^\omega 2$ will be of the form

$$\{f\} \cup \bigcup \{D_{f \upharpoonright n} : n \geq m\},$$

where $m \in \omega$ and $D_{f \upharpoonright n}$ is a dense (hence, as X is submaximal, open) subset of

$U_{f \upharpoonright n}$ for $m \leq n < \omega$. It is easy to see that Y is T_2 , and Y is separable because ω is dense in it.

Now, assume that $D \subset Y$ is dense. Then $D \cap X$ is also dense hence open in X , and similarly $D \cap U_\sigma$ is dense open in U_σ for each $\sigma \in 2^{<\omega}$. So for each $f \in D$ the set $\{f\} \cup \bigcup \{D \cap U_{f \upharpoonright n} : n \geq 0\} \subset D$ is a basic neighbourhood of f , showing that D is open in Y . \square

In 1967 Ceder and Pearson, [9], raised the question whether an ω -resolvable space is necessarily maximally resolvable? El'kin, [12], constructed a T_1 counterexample to this question, and then Malykhin, [28], produced a crowded hereditarily resolvable T_1 space (that is clearly ω -resolvable) which is not maximally resolvable. Eckertson, [11], and later Hu, [20], gave Tychonov counterexamples but not in ZFC: Eckertson's construction used a measurable cardinal, while Hu applied the assumption $2^\omega = 2^{\omega_1}$. Whether one could find a Tychonov counterexample to the Ceder-Pearson problem in ZFC was repeatedly asked as recently as in [7] and [8].

Our next theorem gives a whole class of 0-dimensional T_2 (hence Tychonov) counterexamples to the Ceder-Pearson problem in ZFC. Quite naturally, they involve applications of our main theorem 2.13 where the family of dense sets \mathcal{D} forms a partition of the underlying set.

Recall that any application of theorem 2.13 yields a dense NODEC subspace X of some Cantor cube $D(2)^{2^\kappa}$ with the extra properties

$$|X| = \text{nwd}(X) = \Delta(X) = \kappa.$$

From now on, we shall call any space having all these properties a $\mathcal{C}(\kappa)$ -space. Of course, any $\mathcal{C}(\kappa)$ -space is zero-dimensional T_2 and therefore Tychonov. Finally, with the intention to use lemma 2.8, we recall that any $\mathcal{C}(\kappa)$ -space X , being dense in a Cantor cube, is CCC, i. e. satisfies $\widehat{c}(X) = \omega_1$.

Theorem 2.18. *For any two infinite cardinals $\mu < \kappa$ there is a $\mathcal{C}(\kappa)$ -space X that is the disjoint union of μ dense submaximal subspaces but is not μ^+ -almost-resolvable. (Of course, X is then μ -resolvable but not μ^+ -resolvable, hence not maximally resolvable.)*

Proof. Using fact 2.12 we can easily find a μ -partition $\langle D_i : i < \mu \rangle$ and a family of 2-partitions $\mathbb{B} = \{ \langle B_\xi^0, B_\xi^1 \rangle : \xi < 2^\kappa \}$ of κ such that for each $i < \mu$ and $\varepsilon \in \text{Fn}(2^\kappa, 2)$ we have

$$| D_i \cap \mathbb{B}[\varepsilon] | = \kappa.$$

We may then apply theorem 2.13 to this \mathbb{B} and the family $\mathcal{D} = \{ D_i : i < \mu \}$ to get a collection \mathbb{C} of 2-partitions of κ satisfying 2.13(1)-(4). We claim that the space $X_{\mathbb{C}}$ is as required.

Firstly, as the members of \mathcal{D} partition κ and $X_{\mathbb{C}}$ is NODEC, lemma 2.7 implies that each $D_i \in \mathcal{D}$ is a submaximal dense subspace of $X_{\mathbb{C}}$.

Secondly, since $X_{\mathbb{C}}$ is CCC and $|\mathcal{D}| = \mu \geq \omega$, lemma 2.8 implies that $X_{\mathbb{C}}$ is not μ^+ -almost-resolvable.

□

Theorem 2.18 talks about infinite cardinals, and with good reason; it has been long known that for any finite n there are say countable zero-dimensional spaces

that are n -resolvable but not $(n + 1)$ -resolvable. In connection with this, Eckertson asked in [11, Question 4.5] the following question: *Does there exist for each infinite cardinal κ and for each natural number $n \geq 1$ a Tychonov space X with $|X| = \Delta(X) = \kappa$ such that X is n -resolvable but X contains no $(n + 1)$ -resolvable subspaces?* Li Feng, [15], gave a positive answer to this question and the following corollary of 2.18 improves his result. Our example is a $\mathcal{C}(\kappa)$ -space that is the disjoint union of n dense submaximal subspaces.

Corollary 2.19. *For each cardinal $\kappa \geq \omega$ and each natural number $n \geq 1$ there is a $\mathcal{C}(\kappa)$ -space Y which is the disjoint union of n dense submaximal subspaces. Then Y , automatically, does not contain any $(n + 1)$ -resolvable subspaces.*

Proof. Consider the $\mathcal{C}(\kappa)$ -space X given by theorem 2.18 for any fixed pair of cardinals $\mu < \kappa$ and then set $Y = \bigcup\{D_i : i < n\}$. Here each subspace D_i of Y is submaximal and therefore HI. Consequently, every subspace of Y can be written as the union of at most n HI subspaces. By [21, lemma 2], no such space can be $(n + 1)$ -resolvable, hence Y contains no $(n + 1)$ -resolvable subspaces.

□

Another question that can be raised concerning theorem 2.18 is whether it could be extended to apply to all infinite cardinals instead of just the successors μ^+ . It is actually known that the answer to this question is negative.

Indeed, Illanes, and later Bashkara Rao proved the following two “compactness”-type results on λ -resolvability, for cardinals λ of countable cofinality.

Theorem (Illanes, [21]). *If a topological space X is $(< \omega)$ -resolvable then X is ω -resolvable.*

Theorem (Bhaskara Rao, [3]). *If λ is a singular cardinal with $\text{cf}(\lambda) = \omega$ and X is any topological space that is $(< \lambda)$ -resolvable then X is λ -resolvable.*

In contrast to these, our next result, theorem 2.21, implies that no such compactness-phenomenon is valid for uncountable regular limit (that is inaccessible) cardinals. However, the following intriguing problem remains open.

Problem 2.20. *Assume that λ is a singular cardinal with $\text{cf}(\lambda) > \omega$ and X is a topological space that is $(< \lambda)$ -resolvable. Is it true then that X is also λ -resolvable?*

Theorem 2.21 may be viewed as an extension of 2.18 from successors to all uncountable regular cardinals, providing counterexamples to the Ceder-Pearson problem in further cases. However, the spaces obtained here are quite different from the ones constructed in 2.18 because they are *hereditarily resolvable*.

Theorem 2.21. *For any two cardinals κ and λ with $\omega < \text{cf}(\lambda) = \lambda \leq \kappa$ there is a $\mathcal{C}(\kappa)$ -space that is not λ -almost-resolvable (and hence not λ -resolvable) and still it is hereditarily μ -resolvable for all $\mu < \lambda$.*

Proof. Let us fix the sequence $\vec{\lambda} = \langle \lambda_\zeta : \zeta < \lambda \rangle$ by setting $\lambda_\zeta = \rho$ for each $\zeta < \lambda$ if $\lambda = \rho^+$ is a successor and by putting $\lambda_\zeta = \omega_\zeta$ for $\zeta < \lambda$ if λ is a limit cardinal (note that $\lambda = \omega_\lambda$ in the latter case).

Using fact 2.12 we can find two families of partitions

$$\mathbb{D} = \{\langle D_\zeta^i : i < \lambda_\zeta \rangle : \zeta < \lambda\} \quad \text{and} \quad \mathbb{B} = \{\langle B_\xi^0, B_\xi^1 : \cdot \rangle : \xi < 2^\kappa\}$$

of κ such that $\mathbb{D} \cup \mathbb{B}$ is κ -independent, i. e. $|\mathbb{D}[\eta] \cap \mathbb{B}[\varepsilon]| = \kappa$ whenever $\eta \in \mathbb{FIN}(\vec{\lambda})$ and $\varepsilon \in \text{Fn}(2^\kappa, 2)$. Then

$$\mathcal{D} = \{\mathbb{D}[\eta] : \eta \in \mathbb{FIN}(\vec{\lambda})\}$$

is a family of κ -dense sets in the space $X_{\mathbb{B}}$, hence we can apply theorem 2.13 with \mathbb{B} and \mathcal{D} to get a family \mathcal{C} of 2-partitions of κ satisfying 2.13(1)–(4). We shall show that the $\mathcal{C}(\kappa)$ -space $X_{\mathcal{C}}$ is as required.

Claim 2.21.1. *For every family $\mathcal{E} \in [\mathcal{D}]^\lambda$ there is $\mathcal{F} \in [\mathcal{E}]^\lambda$ such that $D \cap D' \in \mathcal{D}$ (and hence is dense in $X_{\mathcal{C}}$) whenever $\{D, D'\} \in [\mathcal{F}]^2$.*

Proof. We can write $\mathcal{E} = \{\mathbb{D}[\eta_\gamma] : \gamma < \lambda\}$. Since $\lambda = \text{cf}(\lambda) > \omega$ we can find $K \in [\lambda]^\lambda$ such that $\{\text{dom}(\eta_\gamma) : \gamma \in K\}$ forms a Δ -system with kernel K^* . Then $\prod_{i \in K^*} \lambda_i < \lambda$, hence, as λ is regular, there are a set $I \in [K]^\lambda$ and a fixed finite function $\eta \in \prod_{i \in K^*} \lambda_i \subset \mathbb{FIN}(\vec{\lambda})$ such that $\eta_\gamma \upharpoonright K^* = \eta$ for each $\gamma \in I$.

But then $\mathcal{F} = \{\mathbb{D}[\eta_\gamma] : \gamma \in I\}$ is as required: for any $\{\gamma, \delta\} \in [I]^2$ we have $\eta_\gamma \cup \eta_\delta \in \mathbb{FIN}(\vec{\lambda})$, consequently

$$\mathbb{D}[\eta_\gamma] \cap \mathbb{D}[\eta_\delta] = \mathbb{D}[\eta_\gamma \cup \eta_\delta] \in \mathcal{D}.$$

□_{2.21.1}

Now, since $\widehat{c}(X_{\mathcal{C}}) = \omega_1$ and the above claim holds we can apply lemma 2.8 to conclude that $X_{\mathcal{C}}$ is not λ -almost-resolvable.

Let us now fix $\mu < \lambda$. We first show that every $\mathbb{D}[\eta] \in \mathcal{D}$ is μ -resolvable. Indeed, choose $\zeta \in \lambda \setminus \text{dom } \eta$ with $\lambda_\zeta \geq \mu$. Clearly, then the family $\{\mathbb{D}[\eta \cup \{\langle \zeta, \gamma \rangle\}] : \gamma < \lambda_\zeta\}$ forms a partition of $\mathbb{D}[\eta]$ into $\lambda_\zeta \geq \mu$ many dense subsets.

Since $X_{\mathbb{C}}$ is NODEC and \mathcal{D} -forced, any crowded subspace S of $X_{\mathbb{C}}$ is somewhere dense. Consequently, lemma 2.10 implies that $X_{\mathbb{C}}$ is hereditarily μ -resolvable.

□_{2.21}

Remark . It is well-known that any dense subspace of the Cantor cube $D(2)^\lambda$ has weight (even π -weight) equal to λ . Consequently, any $\mathcal{C}(\kappa)$ -space (that is, by definition, of cardinality κ) has maximum possible weight, that is 2^κ . Now, ZFC counterexamples to the Ceder-Pearson problem are naturally expected to have this property. Indeed, for instance the forcing axiom BACH (see e.g. [35]) implies that every topological space X with $|X| = \Delta(X) = \omega_1$ and $\pi w(X) < 2^{\omega_1}$ is ω_1 -resolvable. Consequently, under BACH, any ω -resolvable space X satisfying $|X| = \omega_1$ and $\pi w(X) < 2^{\omega_1}$ is maximally resolvable.

By [21, Lemma 4], any topological space that is not ω -resolvable contains a HI somewhere dense subspace. Theorem 2.21 shows that this badly fails if ω is replaced by an uncountable cardinal.

Again by [21, Lemma 4], if a space X can be partitioned into finitely many dense HI subspaces, then the number of pieces is uniquely determined. It follows from our next result, theorem 2.22 below, that this is not the case for infinite partitions. In fact, for every infinite cardinal κ there is a $\mathcal{C}(\kappa)$ -space that can be simultaneously partitioned into λ many dense submaximal (and so HI) subspaces

for all infinite $\lambda \leq \kappa$.

Theorem 2.22 also gives an affirmative answer to the following question of Eckertson, raised in [11, 3.4 and 3.6]: *Does there exist, for each cardinal μ , a μ^+ -resolvable space that can be partitioned into μ -many dense HI subspaces?*

The proof of theorem 2.22 will require an even more delicate choice of the family of dense sets \mathcal{D} than the one we used in the proof of 2.21.

Theorem 2.22. *For each infinite cardinal κ there is a $\mathcal{C}(\kappa)$ -space that can be simultaneously partitioned into countably many dense hereditarily κ -resolvable subspaces and also into μ many dense submaximal (and therefore HI) subspaces for all infinite $\mu \leq \kappa$.*

Proof. Let us start by setting $\lambda_0 = \omega$, $\lambda_1 = \kappa$, and $\vec{\lambda} = \langle \lambda_i : i < 2 \rangle$, moreover $\vec{\kappa} = \langle \kappa_n : n < \omega \rangle$, where $\kappa_0 = \omega$ and $\kappa_n = \kappa$ for $1 \leq n < \omega$.

By fact 2.12 there are three families of partitions of κ , say

$$\mathbb{B} = \{ \langle B_\zeta^i : i < 2 \rangle : \zeta < 2^\kappa \},$$

$$\mathbb{E} = \{ \langle E_n^j : j < \kappa_n \rangle : n < \omega \},$$

and

$$\mathbb{F} = \{ \langle F_\ell^k : k < \lambda_\ell \rangle : \ell < 2 \},$$

such that $\mathbb{B} \cup \mathbb{E} \cup \mathbb{F}$ is κ -independent, i.e. for each $\varepsilon \in \text{Fn}(2^\kappa, 2)$, $\eta \in \text{FIN}(\vec{\kappa})$, and $\rho \in \text{FIN}(\vec{\lambda})$ we have

$$(\dagger) \quad |\mathbb{B}[\varepsilon] \cap \mathbb{E}[\eta] \cap \mathbb{F}[\rho]| = \kappa.$$

Of course, (\dagger) implies that all sets of the form $\mathbb{E}[\eta] \cap \mathbb{F}[\rho]$ are κ -dense in $X_{\mathbb{B}}$, however the family \mathcal{D} of κ -dense sets that we need will be defined in a more complicated way.

To start with, let us write $\mathcal{F}_\ell = \{F_\ell^k : k < \lambda_\ell\}$ for $\ell < 2$ and then set

$$\mathcal{D}_{\mathbb{E}} = \{\mathbb{E}[\eta] : \eta \in \text{FIN}(\vec{\kappa})\}$$

and

$$\mathcal{D}_{\mathbb{F}} = \mathcal{F}_0 \cup \mathcal{F}_1 = \{F_\ell^k : \ell < 2, \quad k < \lambda_\ell\}.$$

Next let

$$\mathcal{D}_{\mathbb{E},\mathbb{F}} = \{E \setminus \cup \mathcal{F} : E \in \mathcal{D}_{\mathbb{E}}, \quad \mathcal{F} \in [\mathcal{D}_{\mathbb{F}}]^{<\omega}\}$$

and

$$\mathcal{D}_{\mathbb{F},\mathbb{E}} = \{F_\ell^k \setminus ((\cup \mathcal{E}) \cup (\cup \mathcal{F})) : F_\ell^k \in \mathcal{D}_{\mathbb{F}}, \mathcal{E} \in [\mathcal{D}_{\mathbb{E}}]^{<\omega}, \mathcal{F} \in [\mathcal{F}_{1-\ell}]^{<\omega}\}.$$

Finally, we set

$$\mathcal{D} = \mathcal{D}_{\mathbb{E},\mathbb{F}} \cup \mathcal{D}_{\mathbb{F},\mathbb{E}}.$$

Every element of \mathcal{D} contains some (in fact, infinitely many) sets of the form $\mathbb{E}[\eta] \cap \mathbb{F}[\rho]$ and so is κ -dense in $X_{\mathbb{B}}$ by (\dagger) .

Now we may apply theorem 2.13 with \mathbb{B} and \mathcal{D} to obtain a family of partitions \mathbb{C} of κ that satisfies 2.13 (1) - (4). We shall show that $X_{\mathbb{C}}$ is as required.

Claim 2.22.1. *$E \cap F$ is nowhere dense in $X_{\mathbb{C}}$ whenever $E \in \mathcal{D}_{\mathbb{E}}$ and $F \in \mathcal{D}_{\mathbb{F}}$.*

Proof. According to 2.6 it suffices to show that $D \setminus (E \cap F)$ includes an element of \mathcal{D} whenever $D \in \mathcal{D}$.

Now, if $D = E' \setminus \cup \mathcal{F} \in \mathcal{D}_{\mathbb{E}, \mathbb{F}}$ then

$$D \setminus (E \cap F) \supset E' \setminus (\cup(\mathcal{F} \cup \{F\})) \in \mathcal{D}_{\mathbb{E}, \mathbb{F}}.$$

If, on the other hand, $D = F_\ell^k \setminus ((\cup \mathcal{E}) \cup (\cup \mathcal{F})) \in \mathcal{D}_{\mathbb{F}, \mathbb{E}}$ then

$$D \setminus (E \cap F) \supset F_\ell^k \setminus ((\cup(\mathcal{E} \cup \{E\})) \cup (\cup \mathcal{F})) \in \mathcal{D}_{\mathbb{F}, \mathbb{E}}.$$

□_{2.22.1}

Claim 2.22.2. $F \cap F'$ is nowhere dense in $X_{\mathbb{C}}$ for all $\{F, F'\} \in [\mathcal{D}_{\mathbb{F}}]^2$.

Proof. Again, by 2.6, it is enough to show that $D \setminus (F \cap F')$ includes an element of \mathcal{D} for each $D \in \mathcal{D}$.

If $D = E \setminus \cup \mathcal{F} \in \mathcal{D}_{\mathbb{E}, \mathbb{F}}$ then

$$D \setminus (F \cap F') \supset E \setminus (\cup(\mathcal{F} \cup \{F\})) \in \mathcal{D}_{\mathbb{E}, \mathbb{F}}.$$

If $D = F_\ell^k \setminus ((\cup \mathcal{E}) \cup (\cup \mathcal{F})) \in \mathcal{D}_{\mathbb{F}, \mathbb{E}}$ and $F \cap F' \neq \emptyset$ then we can assume that $F \in \mathcal{F}_\ell$ and $F' \in \mathcal{F}_{1-\ell}$. But then we have

$$D \setminus (F \cap F') \supset F_\ell^k \setminus ((\cup \mathcal{E}) \cup (\cup(\mathcal{F} \cup \{F'\}))) \in \mathcal{D}_{\mathbb{F}, \mathbb{E}}.$$

□_{2.22.2}

Claim 2.22.3. Every $D \in \mathcal{D}_{\mathbb{E}, \mathbb{F}}$ is κ -resolvable.

Proof. Let $D = E \setminus \cup \mathcal{F}$. Without loss of generality we can assume that $E = \mathbb{E}[\eta]$ with $\text{dom } \eta = n \in \omega \setminus \{0\}$. But then D is the disjoint union of the $\kappa_n = \kappa$ many dense sets

$$\{\mathbb{E}[\eta \cup \{(n, \zeta)\}] \setminus \cup \mathcal{F} : \zeta < \kappa\}.$$

□_{2.22.3}

Claim 2.22.4. E_0^i is hereditarily κ -resolvable for each $i < \omega = \kappa_0$.

Proof. Let us note first of all that for any

$$D = F \setminus ((\cup \mathcal{E}) \cup (\cup \mathcal{F})) \in \mathcal{D}_{\mathbb{F}, \mathbb{E}}$$

we have $E_0^i \cap D \subset E_0^i \cap F \in \mathcal{N}(X_{\mathbb{C}})$ by claim 2.22.1.

Now, let S be any crowded subspace of E_0^i . Since $X_{\mathbb{C}}$ is NODEC and \mathcal{D} -forced, by lemma 2.10 there is a partial $(\mathcal{D}, X_{\mathbb{C}})$ -mosaic

$$M = \bigcup \{V \cap D_V : V \in \mathcal{V}\} \subset S$$

that is dense in S . By our above remark, we must have $D_V \in \mathcal{D}_{\mathbb{E}, \mathbb{F}}$ whenever $V \in \mathcal{V}$, consequently M and hence S is κ -resolvable by claim 2.22.3 and fact 2.9.

□_{2.22.4}

We have thus concluded that $\{E_0^i : i < \omega\}$ partitions $X_{\mathbb{C}}$ into countably many hereditarily κ -resolvable dense subspaces.

Claim 2.22.5. $F_\ell^k \subset X_{\mathbb{C}}$ is submaximal for all $\ell < 2$ and $k < \lambda_\ell$.

Proof. Since $X_{\mathbb{C}}$ is NODEC, so is its dense subspace F_{ℓ}^k , hence it suffices to show that F_{ℓ}^k is OHI. By lemma 2.7, this will follow if we can show that for each $D \in \mathcal{D}$ either $F_{\ell}^k \cap D$ or $F_{\ell}^k \setminus D$ is nowhere dense in $X_{\mathbb{C}}$.

Case 1. $D = E \setminus \cup \mathcal{F} \in \mathcal{D}_{\mathbb{E}, \mathbb{F}}$.

Then $D \cap F_{\ell}^k \subset E \cap F_{\ell}^k \in \mathcal{N}(X_{\mathbb{C}})$ by claim 2.22.1.

Case 2. $D = F' \setminus ((\cup \mathcal{E}) \cup (\cup \mathcal{F})) \in \mathcal{D}_{\mathbb{F}, \mathbb{E}}$.

If $F' \neq F_{\ell}^k$ then $F_{\ell}^k \cap D \subset F_{\ell}^k \cap F' \in \mathcal{N}(X_{\mathbb{C}})$ by claim 2.22.2. Thus we may assume that $F' = F_{\ell}^k$ and hence $F_{\ell}^k \notin \mathcal{F}$ because $\mathcal{F} \subset \mathcal{F}_{1-\ell}$. But then

$$\begin{aligned} F_{\ell}^k \setminus D &= F_{\ell}^k \setminus (F_{\ell}^k \setminus ((\cup \mathcal{E}) \cup (\cup \mathcal{F}))) = \\ &= F_{\ell}^k \cap ((\cup \mathcal{E}) \cup (\cup \mathcal{F})) = \cup_{E \in \mathcal{E}} (F_{\ell}^k \cap E) \cup \cup_{F \in \mathcal{F}} (F \cap F_{\ell}^k), \end{aligned}$$

where each $F_{\ell}^k \cap E$ is nowhere dense by claim 2.22.1 and each $F \cap F_{\ell}^k$ is nowhere dense by claim 2.22.2, i.e. $F_{\ell}^k \setminus D \in \mathcal{N}(X_{\mathbb{C}})$. $\square_{2.22.5}$

Claim 2.22.5 implies that $X_{\mathbb{C}}$ can be partitioned into μ many dense submaximal subspaces for both $\mu = \omega$ and $\mu = \kappa$. Since $\mathcal{C}(\kappa)$ -spaces are CCC, it follows from theorem 2.23 below that this is also valid for all μ with $\omega < \mu < \kappa$. $\square_{2.22}$

The following result is somewhat different from the others in that it has no relevance to \mathcal{D} -forced spaces. Still we decided to include it here not only because it makes the proof of theorem 2.22 simpler but also because it seems to have independent interest.

Theorem 2.23. *Let $\omega \leq \lambda < \mu < \kappa$ be cardinals and X be a topological space with $c(X) \leq \mu$. If X can be partitioned into both λ many and κ many dense OHI subspaces then X can also be partitioned into μ many dense OHI subspaces.*

Proof. Let $\langle Y_\sigma : \sigma < \lambda \rangle$ and $\langle Z_\zeta : \zeta < \kappa \rangle$ be two partitions of X into OHI subspaces. For each $\sigma < \lambda$ let

$$\mathcal{U}_\sigma = \{U \subset X : U \text{ is open and there is } I_{\sigma,U} \in [\kappa]^\mu \text{ such that}$$

$$Y_\sigma \cap \cup\{Z_\zeta : \zeta \in I_{\sigma,U}\} \text{ is dense in } U\}.$$

Since $c(X) \leq \mu$ there is $\mathcal{U}_\sigma^* \in [\mathcal{U}_\sigma]^{\leq \mu}$ such that $U_\sigma = \cup \mathcal{U}_\sigma^*$ is dense in $\cup \mathcal{U}_\sigma$.

Clearly, we also have $U_\sigma \in \mathcal{U}_\sigma$. Next we set $V_\sigma = X \setminus \overline{U_\sigma}$ and $Q_\sigma = X \setminus (U_\sigma \cup V_\sigma) = \text{Fr}(U_\sigma)$.

Since $\lambda < \mu$ we can pick $I \in [\kappa]^\mu$ with

$$\cup\{I_{\sigma,U_\sigma} : \sigma < \lambda\} \subset I$$

and then can choose $J \in [\kappa \setminus I]^\lambda$. Let $Z = \cup\{Z_\zeta : \zeta \in I \cup J\}$.

For $\sigma \in \lambda$ let $R_\sigma = Y_\sigma \cap V_\sigma \cap Z$. Since $|I \cup J| = \mu$, it follows from the definition of \mathcal{U}_σ and $V_\sigma = X \setminus \overline{\cup \mathcal{U}_\sigma}$ that

(*) R_σ is nowhere dense in X for each $\sigma < \lambda$.

Let $P_\sigma = (Y_\sigma \cap U_\sigma) \setminus \cup\{Z_\zeta : \zeta \in I_{\sigma,U_\sigma}\}$ for $\sigma < \lambda$. Then P_σ is also nowhere dense because $\cup\{Z_\zeta : \zeta \in I_{\sigma,U_\sigma}\} \cap U_\sigma \cap Y_\sigma$ is dense in U_σ and Y_σ is OHI.

Now let $\{\sigma_\zeta : \zeta \in J\}$ be an enumeration of λ without repetition and for each $\zeta \in J$ set

$$T_\zeta = (Z_\zeta \cap U_{\sigma_\zeta}) \cup ((Y_{\sigma_\zeta} \cap V_{\sigma_\zeta}) \setminus Z).$$

Claim 2.23.1. *Each T_ζ is a dense OHI subspace of X .*

Proof. Z_ζ is dense in U_{σ_ζ} and

$$(Y_{\sigma_\zeta} \cap V_{\sigma_\zeta}) \setminus Z = (Y_{\sigma_\zeta} \cap V_{\sigma_\zeta}) \setminus R_{\sigma_\zeta}$$

is dense in V_{σ_ζ} because Y_{σ_ζ} is dense and $R_{\sigma_\zeta} = Y_{\sigma_\zeta} \cap V_{\sigma_\zeta} \cap Z$ is nowhere dense by (*). Hence T_ζ is dense. T_ζ is OHI because both Z_ζ and Y_{σ_ζ} are. \square

Claim 2.23.2. *The family $\{Z_\xi : \xi \in I\} \cup \{T_\zeta : \zeta \in J\}$ is disjoint.*

Proof. Assume first that $\xi \in I$ and $\zeta \in J$. Then $\xi \neq \zeta$ and hence

$$\begin{aligned} T_\zeta \cap Z_\xi &= ((Z_\zeta \cap U_{\sigma_\zeta}) \cup ((Y_{\sigma_\zeta} \cap V_{\sigma_\zeta}) \setminus Z)) \cap Z_\xi \\ &\subset (Z_\zeta \cap Z_\xi) \cup (Z_\xi \setminus Z) = \emptyset. \end{aligned}$$

Next if $\{\zeta, \xi\} \in [J]^2$, then

$$\begin{aligned} T_\zeta \cap T_\xi &= \\ &= \left((Z_\zeta \cap U_{\sigma_\zeta}) \cup ((Y_{\sigma_\zeta} \cap V_{\sigma_\zeta}) \setminus Z) \right) \cap \left((Z_\xi \cap U_{\sigma_\xi}) \cup ((Y_{\sigma_\xi} \cap V_{\sigma_\xi}) \setminus Z) \right) \subset \\ &\subset (Z_\zeta \cap Z_\xi) \cup (Z_\zeta \setminus Z) \cup (Z_\xi \setminus Z) \cup (Y_{\sigma_\zeta} \cap Y_{\sigma_\xi}) = \emptyset. \end{aligned}$$

\square

Thus we would be finished if we could prove that

$$\{Z_\xi : \xi \in I\} \cup \{T_\zeta : \zeta \in J\}$$

covers X . However, we can only prove the following weaker statement.

Claim 2.23.3.

$$X = \bigcup \{Z_\xi : \xi \in I\} \cup \bigcup \{T_\zeta : \zeta \in J\} \cup \bigcup \{P_\sigma \cup Q_\sigma \cup R_\sigma : \sigma < \lambda\}.$$

Proof. Let $x \in X$ be any point then there is a unique $\sigma < \lambda$ with $x \in Y_\sigma$. If $x \notin U_\sigma \cup V_\sigma$ then, by definition, $x \in Q_\sigma$.

So assume now that $x \in U_\sigma$. If $x \notin \bigcup \{Z_\zeta : \zeta \in I_{\sigma, U_\sigma}\}$ then $x \in P_\sigma$. Otherwise $x \in Z_\zeta$ for some $\zeta \in I_{\sigma, U_\sigma} \subset I$.

Finally, assume that $x \in V_\sigma$ and let $\zeta \in J$ with $\sigma_\zeta = \sigma$. Now, if $x \notin Z$ then $x \in T_\zeta$ and if $x \in Z$ then $x \in R_\sigma$. \square

The pairwise disjoint dense OHI subspaces $\{Z_\xi : \xi \in I\} \cup \{T_\zeta : \zeta \in J\}$ thus cover X apart from the nowhere dense sets $P_\sigma \cup Q_\sigma \cup R_\sigma$ for $\sigma < \lambda$. But then, using the obvious fact that the union of a dense OHI subspace with any nowhere dense set is OHI, the latter can be simply “absorbed” by the former, and thus a partition of X into μ many dense OHI subspaces can be produced. $\square_{2.23}$

2.4 Applications to extraresolvability

In [8] Comfort and Hu investigated the following question: *Are maximally resolvable spaces (strongly) extraresolvable?* They presented several counterexamples, but the following question was left open (see [8, Discussion 1.4]): *Is there a maximally resolvable Tychonov space X with $|X| = \text{nwd}(X)$ such that X is not extraresolvable?* Using our main theorem 2.13 we can give an affirmative answer to this question in ZFC. Recall that if X is a $\mathcal{C}(\kappa)$ -space then $|X| = \text{nwd}(X) = \kappa$.

Theorem 2.24. *For every infinite cardinal κ there is a $\mathcal{C}(\kappa)$ -space that is hereditarily κ -resolvable (and hence maximally resolvable) but not extraresolvable.*

Proof. Let $\vec{\kappa} = \langle \kappa, \kappa, \dots \rangle$ be the constant κ sequence of length ω . By fact 2.12 there are a countable family $\mathbb{D} = \{ \langle D_m^i : i < \kappa \rangle : m < \omega \}$ of κ -partitions of κ and a family $\mathbb{B} = \{ \langle B_\xi^0, B_\xi^1 : \rangle : \xi < 2^\kappa \}$ of 2-partitions of κ such that $\mathbb{B} \cup \mathbb{D}$ is κ -independent, that is for each $\eta \in \mathbb{FIN}(\vec{\kappa}) = \text{Fn}(\omega, \kappa)$ and $\varepsilon \in \text{Fn}(2^\kappa, 2)$ we have

$$| \mathbb{D}[\eta] \cap \mathbb{B}[\varepsilon] | = \kappa.$$

Now let

$$\mathcal{D} = \{ \mathbb{D}[\eta] : \eta \in \mathbb{FIN}(\vec{\kappa}) \}$$

and apply theorem 2.13 to \mathbb{B} and \mathcal{D} to get a family \mathbb{C} of 2-partitions of κ satisfying 2.13 (1) - (4).

Since $|\mathcal{D}| = \kappa$ and $\widehat{c}(X_{\mathbb{C}}) = \omega_1$, it follows from lemma 2.8 that $X_{\mathbb{C}}$ is not κ^+ -almost-resolvable (= almost-resolvable).

Next, if $\mathbb{D}[\eta] \in \mathcal{D}$ then $\{ \mathbb{D}[\eta \cap \langle \zeta \rangle] : \zeta < \kappa \}$ partitions $\mathbb{D}[\eta]$ into κ many dense sets, i.e. $\mathbb{D}[\eta]$ is κ -resolvable. Hence, by lemma 2.10, $X_{\mathbb{C}}$ is hereditarily κ -resolvable. \square

Our next two results are natural analogues of theorems 2.18 and 2.21 with μ -resolvability replaced by μ -extraresolvability. Before formulating them, however, we need a new piece of notation.

Definition 2.25. Given a family $\mathbb{D} = \{\langle D_\xi^0, D_\xi^1 \rangle : \xi \in \rho\}$ of 2-partitions of a cardinal κ we set

$$\mathcal{I}(\mathbb{D}) = \{D_\zeta^0 \setminus \cup_{\xi \in \Xi} D_\xi^0 : \zeta \in \rho \wedge \Xi \in [\rho \setminus \{\zeta\}]^{<\omega}\}.$$

Theorem 2.26. *For any infinite cardinals $\kappa \leq \lambda \leq 2^\kappa$ there is a λ -extraresolvable $\mathcal{C}(\kappa)$ -space X that is not λ^+ -almost-resolvable. Moreover, every crowded subspace of X has a dense submaximal subspace.*

Proof. By fact 2.12 there are families of 2-partitions of κ , say $\mathbb{D} = \{\langle D_\zeta^0, D_\zeta^1 \rangle : \zeta < \lambda\}$ and $\mathbb{B} = \{\langle B_\xi^0, B_\xi^1 \rangle : \xi < 2^\kappa\}$, such that $\mathbb{B} \cup \mathbb{D}$ is κ -independent, i. e. $|\mathbb{D}[\eta] \cap \mathbb{B}[\varepsilon]| = \kappa$ for all $\eta \in \text{Fn}(\lambda, 2)$ and $\varepsilon \in \text{Fn}(2^\kappa, 2)$.

Then $\mathcal{D} = \mathcal{I}(\mathbb{D})$ is a family of κ -dense subsets of $X_{\mathbb{B}}$, hence we can apply the main theorem 2.13 to \mathbb{B} and \mathcal{D} to obtain a family of partitions \mathbb{C} satisfying 2.13 (1) - (4). We shall show that $X_{\mathbb{C}}$ is as required.

Claim 2.26.1. $D_\zeta^0 \cap D_\xi^0 \in \mathcal{N}(X_{\mathbb{C}})$ for each pair $\{\zeta, \xi\} \in [\lambda]^2$.

Proof. Write $Y = D_\zeta^0 \cap D_\xi^0$ and $D = D_\nu^0 \setminus \cup_{\eta \in \Xi} D_\eta^0$ be an arbitrary member of \mathcal{D} . We can assume that $\xi \neq \nu$ and so

$$D \setminus Y = (D_\nu^0 \setminus \cup_{\eta \in \Xi} D_\eta^0) \setminus (D_\zeta^0 \cap D_\xi^0) \supset D_\nu^0 \setminus \cup_{\eta \in \Xi \cup \{\xi\}} D_\eta^0 \in \mathcal{D},$$

showing that $D \setminus Y$ is dense in $X_{\mathbb{C}}$. Hence, by lemma 2.6, Y is nowhere dense in $X_{\mathbb{C}}$. \square

Thus the family $\{D_\xi^0 : \xi \in \lambda\}$ witnesses that $X_{\mathbb{C}}$ is λ -almost-resolvable. On the other hand, since $|\mathcal{D}| = \lambda$ and $c(X_{\mathbb{C}}) = \omega$, lemma 2.8 implies that $X_{\mathbb{C}}$ is not λ^+ -almost-resolvable.

Claim 2.26.2. *Every $S \in \mathcal{D}$ is a submaximal subspace of $X_{\mathbb{C}}$.*

Proof. Let $S = D_{\nu}^0 \setminus \cup_{\eta \in \Xi} D_{\eta}$, moreover $D = D_{\mu}^0 \setminus \cup_{\eta \in \Psi} D_{\eta}^0$ be an arbitrary member of \mathcal{D} . If $\nu = \mu$ then, by claim 2.26.1,

$$S \setminus D = (D_{\nu}^0 \setminus \cup_{\eta \in \Xi} D_{\eta}^0) \setminus (D_{\nu}^0 \setminus \cup_{\eta \in \Psi} D_{\eta}^0) \subset \bigcup_{\eta \in \Psi} D_{\nu}^0 \cap D_{\eta}^0 \in \mathcal{N}(X_{\mathbb{C}})$$

and so $S \subset^* D$. If, on the other hand, $\nu \neq \mu$ then we have

$$S \cap D = (D_{\nu}^0 \setminus \cup_{\eta \in \Xi} D_{\eta}^0) \cap (D_{\nu}^0 \setminus \cup_{\eta \in \Psi} D_{\eta}^0) \subset D_{\nu}^0 \cap D_{\mu}^0 \in \mathcal{N}(X_{\mathbb{C}})$$

by claim 2.26.1 again, consequently $S \cap D =^* \emptyset$. Thus S is OHI by lemma 2.7, and since $X_{\mathbb{C}}$ is NODEC, S is even submaximal. \square

Claim 2.26.2 clearly implies that all \mathcal{D} -pieces and hence all partial \mathcal{D} -mosaics are submaximal subspaces of $X_{\mathbb{C}}$. But $X_{\mathbb{C}}$ is \mathcal{D} -forced and NODEC, and therefore, by lemma 1.10, every crowded subspace of $X_{\mathbb{C}}$ includes a partial \mathcal{D} -mosaic as a dense subspace. $\square_{2.26}$

Let us remark that theorem 2.26 makes sense, and remains valid, for $\lambda < \kappa$ as well. However, in this case theorem 2.18 yields a stronger result. This is the reason why we only formulated it for $\lambda \geq \kappa$. This remark also applies to our following result that implies an analogue of theorem 2.21 for μ -extraresolvability instead of μ -resolvability.

Theorem 2.27. *Let $\kappa < \lambda = \text{cf}(\lambda) \leq (2^{\kappa})^{+}$ be infinite cardinals. Then there is a $\mathcal{C}(\kappa)$ -space that is*

1. hereditarily κ -resolvable,
2. hereditarily μ -almost-resolvable for all $\mu < \lambda$,
3. not λ -almost-resolvable.

Proof. Similarly as in the proof of 2.21, let the sequence $\vec{\lambda} = \langle \lambda_\zeta : \zeta < \lambda \rangle$ be given by $\lambda_\zeta = \omega_\zeta$ if λ is a limit (hence inaccessible) cardinal, and let $\lambda_\zeta = \rho$ for each $\zeta < \lambda$ if $\lambda = \rho^+$ is a successor.

Using fact 2.12 again, we can find the following two types of families of 2-partitions of κ :

$$\mathbb{B} = \{ \langle B_\xi^0, B_\xi^1 \rangle : \xi < 2^\kappa \}$$

and

$$\mathbb{D}_\zeta = \{ \langle D_{\zeta,\nu}^0, D_{\zeta,\nu}^1 \rangle : \nu < \lambda_\zeta \}$$

for all $\zeta < \lambda$, moreover a countable family

$$\mathbb{G} = \{ \langle G_n^i : i < \kappa \rangle : n < \omega \}$$

of κ -partitions of κ such that $\mathbb{B} \cup \bigcup_{\zeta < \lambda} \mathbb{D}_\zeta \cup \mathbb{G}$ is κ -independent.

Now let \mathcal{D} be the family of all sets of the form $\bigcap_{i < n} E_i \cap \mathbb{G}[\eta]$ where $n < \omega$ and $E_i \in \mathcal{I}(\mathbb{D}_{\zeta_i})$ with all the ζ_i distinct, moreover $\eta \in \text{Fn}(\omega, \omega)$. It is easy to see that \mathcal{D} is a family of κ -dense sets in $X_{\mathbb{B}}$, so we may apply theorem 2.13 with \mathbb{B} and \mathcal{D} to get a family of partitions \mathbb{C} satisfying 2.13 (1) - (4). We claim that $X_{\mathbb{C}}$ is as required.

Indeed, as we have already seen many times, the $\mathbb{G}[\eta]$ components of the elements of \mathcal{D} can be used to show that every $D \in \mathcal{D}$ is κ -resolvable. But then, as

$X_{\mathbb{C}}$ is both \mathcal{D} -forced and NODEC, every crowded subspace of $X_{\mathbb{C}}$ is κ -resolvable by lemma 2.10, hence (1) is proven.

To prove (2), we need the following statement.

Claim 2.27.1. *Assume that $\zeta < \lambda$ and $\{\nu, \nu'\} \in [\lambda_{\zeta}]^2$. Then*

$$Y = D_{\zeta, \nu}^0 \cap D_{\zeta, \nu'}^0 \in \mathcal{N}(X_{\mathbb{C}}).$$

Proof. Let $D = \bigcap_{i < n} E_i \cap G$ be an arbitrary element of \mathcal{D} , where $n \in \omega$, $\{\zeta_i : i < n\} \in [\lambda]^n$ with $E_i \in \mathcal{I}(\mathbb{D}_{\zeta_i})$ for all $i < n$, and $G = \mathbb{G}[\eta]$ for some $\eta \in \text{Fn}(\omega, \omega)$. Our aim is to check that $D \setminus Y$ is dense, hence, by shrinking D if necessary, we may assume that $\zeta_0 = \zeta$ and $E_0 = D_{\zeta, \varphi}^0 \setminus \bigcup_{\xi \in \Psi} D_{\zeta, \xi}^0$. Since $\nu \neq \nu'$ we can assume that $\varphi \neq \nu$. Then

$$\begin{aligned} D \setminus Y &\supset (\bigcap_{i < n} E_i \cap G) \setminus D_{\zeta, \nu}^0 = \\ &= (D_{\zeta, \varphi}^0 \setminus \bigcup_{\xi \in \Psi \cup \{\nu\}} D_{\zeta, \xi}^0) \cap \bigcap_{i=1}^{n-1} E_i \cap G \in \mathcal{D}. \end{aligned}$$

Hence, $D \setminus Y$ is indeed dense and so, by lemma 2.6, Y is nowhere dense in $X_{\mathbb{C}}$. \square

Assume now that $D = \bigcap_{i < n} E_i \cap G$ is again an arbitrary element of \mathcal{D} with $E_i \in \mathcal{I}(\mathbb{D}_{\zeta_i})$ for all $i < n$. By claim 2.27.1, for every ζ that is distinct from all the ζ_i the collection

$$\{D \cap D_{\zeta, \nu}^0 : \nu < \lambda_{\zeta}\}$$

consists of members of \mathcal{D} that have pairwise nowhere dense intersections, hence D is λ_{ζ} -almost-resolvable. Clearly, this implies that D is μ -almost-resolvable for

all $\mu < \lambda$. By lemma 2.10, since $X_{\mathbb{C}}$ is \mathcal{D} -forced and NODEC it follows that $X_{\mathbb{C}}$ is hereditarily μ -extraresolvable for all $\mu < \lambda$ and thus (2) has been established.

Finally, a standard Δ -system and counting argument proves that for each $\mathcal{E} \in [\mathcal{D}]^\lambda$ there is $\mathcal{F} \in [\mathcal{E}]^\lambda$ such that $F \cap F' \in \mathcal{D}$ whenever $\{F, F'\} \in [\mathcal{F}]^2$. Hence, by lemma 2.8, the space $X_{\mathbb{C}}$ is not λ -almost-resolvable, proving (3). \square

Having seen these parallels between resolvability and extraresolvability, it is interesting to note that we do not know if the analogue of Bashkara Rao's "compactness" theorem holds for extraresolvability.

Problem 2.28. *Assume that λ is a singular cardinal with $\text{cf}(\lambda) = \omega$ and the space X is μ -almost-resolvable for all $\mu < \lambda$. Is it true then that X is also λ -almost-resolvable ?*

3 Spaces having small spread

El'kin proved in [12] that, for any cardinal κ , every space may be written as the disjoint union of a hereditarily κ -irresolvable open subset and a κ -resolvable closed subset. As Pavlov observed in the introduction of [31], this statement has the following reformulation.

Lemma 3.1. *A topological space X is κ -resolvable iff every nonempty open subspace of X includes a nonempty κ -resolvable subset, in other words: iff X has a π -network consisting of κ -resolvable subsets.*

For any topological space X we let $\text{ls}(X)$ denote the minimum number of left-separated subspaces needed to cover X . The following lemma is implicit in the proof of [31, Theorem 2.8] and easily follows from the well-known fact that every space has a *dense* left-separated subspace, see e. g. [22, 2.9.c].

Lemma 3.2. *If for each $U \in \tau^*(X)$ we have $\text{ls}(U) \geq \kappa$, that is no nonempty open set in X can be covered by fewer than κ many left separated sets, then X is κ -resolvable.*

Our next lemma generalizes propositions 2.3 and 3.3 from [31]. We believe that our present approach is not only more general but also simpler than that in [31]. To formulate the lemma, we need to introduce a piece of notation.

Given a family of sets \mathcal{A} and a cardinal κ , we denote by $S_\kappa(\mathcal{A})$ the collection of all *disjoint* subfamilies of \mathcal{A} of size less than κ , i. e.

$$S_\kappa(\mathcal{A}) = \{\mathcal{A}' \in [\mathcal{A}]^{<\kappa} : \mathcal{A}' \text{ is disjoint}\}.$$

Lemma 3.3. *Let us be given a topological space X , a dense set $D \subset X$, an infinite cardinal $\kappa \geq |D|$, moreover a family $\mathcal{I} \subset \mathcal{P}(X)$ of subsets of X . If for each $x \in D$ and for any $\mathcal{Y} \in S_\kappa(\mathcal{I})$ there is a set $Z \in \mathcal{I}$ such that $\cup \mathcal{Y} \cap Z = \emptyset$ and $x \in \overline{Z}$ then X is κ -resolvable.*

Proof. Let $\{x_\alpha : \alpha < \kappa\} = D$ be a κ -abundant enumeration of D , that is for any point $x \in D$ we have $a_x = \{\alpha : x_\alpha = x\} \in [\kappa]^\kappa$. By a straightforward transfinite recursion on $\alpha < \kappa$ we may then choose sets $Z_\alpha \in \mathcal{I} \cap \mathcal{P}(X \setminus \cup_{\nu < \alpha} Z_\nu)$ with $x_\alpha \in \overline{Z_\alpha}$ for all $\alpha < \kappa$. (Note that we have $\{Z_\nu : \nu < \alpha\} \in S_\kappa(\mathcal{I})$ along the way.)

For any ordinal $i < \kappa$ and for any point $x \in D$ let α_i^x be the i th element of the set a_x and set

$$D_i = \bigcup \{Z_{\alpha_i^x} : x \in D\}.$$

Then clearly $D \subset \overline{D_i}$, hence $\{D_i : i < \kappa\}$ is a disjoint family of dense sets, witnessing that X is κ -resolvable. \square

As an illustration, note that if $|X| = \Delta(X) = \kappa > \lambda$ and $t(x, X) \leq \lambda$ holds for all points $x \in D$ of a set D which is dense in the space X , then D , X , κ , and $\mathcal{I} = [X]^{\leq \lambda}$ satisfy the conditions of lemma 3.3 and so X is κ -resolvable. Thus we obtain the following result as an immediate corollary of lemma 3.3.

Corollary 3.4. *If $\Delta(X) > \sup\{t(x, X) : x \in D\}$ for some dense set $D \subset X$ then X is maximally resolvable. In particular, if $\Delta(X) > t(X)$ then X is maximally resolvable.*

The second statement is a theorem of Pytkeev from [32].

3.1 Improving Pavlov's result concerning spread

As was mentioned in the abstract, in [31] Pavlov defined $\text{ps}(X)$ as the smallest successor cardinal such that X has no discrete subset of that size. We recall from [22, 1.22] the related definition of $\widehat{\text{s}}(X)$ that is the smallest uncountable cardinal such that X has no discrete subset of that size. Clearly, one has $\widehat{\text{s}}(X) \leq \text{ps}(X)$ and $\widehat{\text{s}}(X) = \text{ps}(X)$ iff $\widehat{\text{s}}(X)$ is a successor. Finally, let us define $\text{rs}(X)$ as the smallest uncountable regular cardinal such that X has no discrete subset of that size. Then we have $\widehat{\text{s}}(X) \leq \text{rs}(X) \leq \text{ps}(X)$ and $\widehat{\text{s}}(X) = \text{rs}(X)$ iff $\widehat{\text{s}}(X)$ is regular.

In [31] it was shown that if a space X satisfies $\Delta(X) > \text{ps}(X)$ then X is maximally (i. e. $\Delta(X)$) resolvable. The aim of this section is to improve this result by showing that the assumption $\Delta(X) > \text{ps}(X)$ can be relaxed to $\Delta(X) \geq \text{rs}(X)$.

Before doing that, however, we have to give an auxiliary result that involves the cardinal function $\text{h}(X)$, or more precisely its "hatted" version $\widehat{\text{h}}(X)$. We recall that $\widehat{\text{h}}(X)$ is the smallest uncountable cardinal such that X has no right separated subset of that size, or equivalently, the smallest uncountable cardinal κ with the property that any family \mathcal{U} of open sets in X has a subfamily \mathcal{V} of size $< \kappa$ such that $\cup \mathcal{V} = \cup \mathcal{U}$, see e. g. [22, 2.9.b].

Lemma 3.5. *If κ is an uncountable regular cardinal and*

$$|X| \geq \kappa \geq \widehat{\text{h}}(X)$$

then X contains a κ -resolvable subspace X^ .*

Proof. We can assume without loss of generality that $X = \langle \kappa, \tau \rangle$. Let us denote by $\text{NS}(\kappa)$ the ideal of non-stationary subsets of κ and set $\mathcal{G} = \{U \in \tau : U \in \text{NS}(\kappa)\}$. Since $\widehat{\text{h}}(X) \leq \kappa$ there is $\mathcal{G}' \in [\mathcal{G}]^{<\kappa}$ with $\cup \mathcal{G}' = \cup \mathcal{G} = G$. Then $G \in \text{NS}(\kappa)$ because the ideal $\text{NS}(\kappa)$ is κ -complete.

Let us now consider the set

$$T = \{x \in \kappa : \exists C_x \subset \kappa \text{ club } (\forall S \subset C_x \text{ if } S \in \text{NS}(\kappa) \text{ then } x \notin \overline{S})\}.$$

Claim 3.5.1. $T \in \text{NS}(\kappa)$.

Assume, on the contrary, that T is stationary in κ . Fix for each $x \in T$ a club C_x as above. Then the diagonal intersection

$$C = \Delta\{C_x : x \in T\}$$

is again club and so $C \cap T$ is stationary in κ as well. We may then choose a set $S \in [C \cap T]^\kappa$ that is non-stationary. But then for each $x \in S$ we have

$$S \setminus (x+1) \subset C \setminus (x+1) \subset C_x,$$

hence by the choice of C_x we have $x \notin \overline{S \setminus (x+1)}$. Consequently, S is right separated in its natural well-ordering, contradicting the assumption $\widehat{h}(X) \leq \kappa$, and so our claim has been verified.

Finally, put $X^* = X \setminus (G \cup T)$ and $\mathcal{I} = \text{NS}(\kappa) \cap \mathcal{P}(X^*)$. Then lemma 3.3 can be applied to the space X^* , with itself as a dense subspace, the cardinal κ , and the family \mathcal{I} . Indeed, for any point $x \in X^*$ and for any non-stationary set $Y \subset X^*$ there is a club set $C \subset X^* \setminus Y$, and then $x \notin T$ implies that $x \in \overline{Z}$ for some non-stationary set $Z \subset C$. (We have, of course, used here that \mathcal{I} is κ -complete.) This shows that X^* is indeed κ -resolvable. \square

We are now ready to formulate and prove the promised improvement of Pavlov's theorem.

Theorem 3.6. *Let X be a space and κ be a regular cardinal such that*

$$\widehat{s}(X) \leq \kappa \leq \Delta(X),$$

then X is κ -resolvable. Consequently, if $\Delta(X) \geq \text{rs}(X)$ holds for a space X then X is maximally resolvable. In particular, any space of countable spread and uncountable dispersion character is maximally resolvable.

Proof. In view of lemma 4.1 it suffices to show that any non-empty open subset G of X includes a κ -resolvable subspace. To this end, note that, trivially, for each $G \in \tau^*(X)$ we have either

$$(i) \text{ ls}(H) \geq \kappa \text{ for all } H \in \tau^*(G),$$

or

$$(ii) \text{ ls}(H) < \kappa \text{ for some } H \in \tau^*(G).$$

In case (i) G itself is κ -resolvable by lemma 3.2. In case (ii) we claim that $\widehat{\text{h}}(H) \leq \kappa$ holds true and therefore H (and hence G) contains a κ -resolvable subset by lemma 3.5. Assume, on the contrary, that $R \subset H$ is right-separated and has cardinality κ . Since $H = \bigcup \{L_\alpha : \alpha < \text{ls}(H)\}$, where the sets L_α are all left-separated, there is an $\alpha < \text{ls}(H) < \kappa$ such that $|R \cap L_\alpha| = \kappa$ because κ is regular. But then the subspace $R \cap L_\alpha$ is both right and left separated, hence (see e. g. [22, 2.12]) it contains a discrete subset of size $|R \cap L_\alpha| = \kappa$, contradicting our assumption that $\widehat{\text{s}}(X) \leq \kappa$.

If $\Delta(X)$ is regular then this immediately yields that X is maximally resolvable, while if $\Delta(X)$ is singular then, as $\text{rs}(X)$ is regular, we have

$$\Delta(X) > \text{rs}(X)^+ \geq \text{ps}(X),$$

hence Pavlov's result [31, 2.9] may be applied to get the second part, of which the third is a special case.

□

It is natural to raise the question if theorem 3.6 could be further improved by replacing $\text{rs}(X)$ with $\widehat{\text{s}}(X)$ in it. Of course, this is really a problem only in the case when

$$\Delta(X) = \widehat{\text{s}}(X) = \lambda$$

is a singular cardinal. Recall now that Hajnal and Juhász proved in [19] (see also [22, 4.2]) that $\widehat{\text{s}}(X)$ can not be singular *strong limit* for a Hausdorff space X . Consequently, the above mentioned strengthening is valid for Hausdorff spaces provided that all singular cardinals are strong limit, in particular if GCH holds.

Corollary 3.7. *Assume that for every (infinite) cardinal κ the power 2^κ is a finite successor of κ (or equivalently, all singular cardinals are strong limit). Then every Hausdorff space X satisfying $\Delta(X) \geq \widehat{\text{s}}(X)$ is maximally resolvable.*

It is also known (see e. g. [22, 4.3]) that $\widehat{\text{s}}(X)$ can not have countable cofinality for a strongly Hausdorff, in particular for a T_3 space X . Hence the first interesting ZFC question that is left open by theorem 3.6 is the following.

Problem 3.8. *Assume that X is a T_3 space satisfying*

$$\widehat{\text{s}}(X) = \Delta(X) = \aleph_{\omega_1}.$$

Is X then (maximally) resolvable?

It is clear that if in theorem 3.6 we have $\Delta(X) = \lambda > \text{rs}(X)$ then the first part may be applied to any regular cardinal κ with $\text{rs}(X) \leq \kappa \leq \lambda$, hence if λ is singular then we obtain that X is $(< \lambda)$ -resolvable without any reference to Pavlov's result. This is of significance because the proof of Pavlov's theorem in the case when $\Delta(X)$ is singular is rather involved. However, if in addition λ has countable cofinality then no reference to Pavlov's proof is needed because of the following result of Bhaskara Rao.

Theorem (Bhaskara Rao, [3]). *If $\text{cf}(\lambda) = \omega$ and the space X is $(< \lambda)$ -resolvable then X is also λ -resolvable.*

The question if the analogous result can be proved for singular cardinals of uncountable cofinality is one of the outstanding open problems in the area of resolvability and was already formulated in [23]. We just repeat it here.

Problem 3.9. *Assume that λ is a singular cardinal with $\text{cf}(\lambda) > \omega$ and the space X is $(< \lambda)$ -resolvable. Is it true then that X is also λ -resolvable?*

We close this section by giving a partial affirmative answer to problem 3.9. At the same time we shall also show how the first part of theorem 3.6 implies the second in case $\Delta(X)$ is singular, thus making our proof of 3.6 self-contained. To do this, we shall first fix some notation.

Definition 3.10. For any space X we let $\mathcal{D}(X)$ denote the family of all dense subsets of X . Next, we set

$$\mathcal{F}(X) = \cup\{\mathcal{D}(U) : U \in \tau^*(X)\};$$

we call the members of $\mathcal{F}(X)$, i. e. dense subsets of (non-empty) open sets, *fat* sets in X .

For a subspace $Y \subset X$ and a cardinal ν we let

$$\mathcal{H}(Y, \nu) = \mathcal{F}(X) \cap [Y]^{\leq \nu},$$

in other words, $\mathcal{H}(Y, \nu)$ is the family of all fat (in X !) subsets of Y of size at most ν . It is easy to see that if $c(X) \leq \nu$ and $\mathcal{H}(Y, \nu)$ is non-empty then there is a member $H(Y, \nu) \in \mathcal{H}(Y, \nu)$ of maximal closure, i. e. such that

$$\overline{H(Y, \nu)} = \overline{\cup \mathcal{H}(Y, \nu)}.$$

(If $\mathcal{H}(Y, \nu)$ is empty then we set $H(Y, \nu) = \emptyset$.) Clearly, if $Y \subset Z \subset X$ and $c(X) \leq \nu$ then we have

$$\overline{H(Y, \nu)} \subset \overline{H(Z, \nu)}.$$

Finally, we define the *local density* $d_0(X)$ of the space X by

$$d_0(X) = \min\{d(U) : U \in \tau^*(X)\}.$$

Clearly, we have

$$d_0(X) = \min\{|A| : A \in \mathcal{F}(X)\} = \min\{\Delta(D) : D \in \mathcal{D}(X)\}.$$

The following result is obvious but very useful.

Lemma 3.11. *Let X be a space and λ a singular cardinal such that every $D \in \mathcal{D}(X)$ is $(< \lambda)$ -resolvable. Then X is λ -resolvable.*

As an immediate consequence of lemma 3.11 and of the first part of theorem 3.6 we obtain that if λ is singular and $s(X) < \lambda \leq d_0(X)$ then X is λ -resolvable. (Of course, here $s(X) < \lambda$ is equivalent with $\widehat{s}(X) < \lambda$ or with $\text{ps}(X) < \lambda$.)

The following lemma shows that, under certain simple and natural conditions, if a space X is not μ -resolvable for some cardinal μ then some open set $V \in \tau^*(X)$ satisfies a condition just slightly weaker than $\mu \leq d_0(V)$.

Lemma 3.12. *Let X and μ be such that $c(X) < \mu \leq \Delta(X)$. Then either X is μ -resolvable or*

- (*) *there is $V \in \tau^*(X)$ such that for each $\kappa < \mu$ there is $T \in [V]^{<\mu}$ with $d_0(V \setminus T) > \kappa$.*

If μ is regular then $V \in \tau^(X)$ and $T \in [V]^{<\mu}$ may even be chosen so that $d_0(V \setminus T) \geq \mu$.*

Proof of lemma 3.12. Let us first consider the case when μ is regular and assume that for all $V \in \tau^*(X)$ and $T \in [V]^{<\mu}$ we have $d_0(V \setminus T) < \mu$. We define pairwise disjoint dense sets $D_\alpha \in \mathcal{D}(X) \cap [X]^{<\mu}$ for $\alpha < \mu$ by transfinite recursion as follows.

Assume that $\{D_\beta : \beta \in \alpha\} \subset \mathcal{D}(X) \cap [X]^{<\mu}$ have already been defined and set $T = \cup\{D_\beta : \beta \in \alpha\}$, then $|T| < \mu$ as μ is regular. Let \mathcal{W} be a maximal disjoint collection of open sets $W \in \tau^*(X)$ such that $d(W \setminus T) < \mu$. By our assumption, then $\cup\mathcal{W}$ is dense in X and hence so is $\cup\{W \setminus T : W \in \mathcal{W}\}$. So if for each $W \in \mathcal{W}$ we fix $D_W \in \mathcal{D}(W \setminus T)$ with $|D_W| < \mu$ then $D_\alpha = \cup\{D_W : W \in \mathcal{W}\}$

is dense in X as well and clearly $|D_\alpha| < \mu$. The family $\{D_\alpha : \alpha < \mu\}$ witnesses that X is μ -resolvable.

So let us assume now that μ is singular and fix a strictly increasing sequence $\langle \mu_\alpha : \alpha < \text{cf}(\mu) \rangle$ of regular cardinals converging to μ with $c(X) \cdot \text{cf}(\mu) < \mu_0$.

We then define a $\text{cf}(\mu) \times \mu$ type matrix $\{A_\xi^\alpha : \alpha < \text{cf}(\mu), \xi < \mu\}$ of pairwise disjoint subsets of X , column by column in $\text{cf}(\mu)$ steps, as follows:

$$X_\alpha = X \setminus \bigcup \{A_\xi^\beta : \beta < \alpha, \xi < \mu\},$$

$$A_\xi^\alpha = H(X_\alpha \setminus \bigcup \{A_\zeta^\alpha : \zeta < \xi\}, \mu_\alpha).$$

Observe that we have $|A_\xi^\alpha| \leq \mu_\alpha$, moreover

$$(\dagger) \quad \overline{A_\xi^\alpha} \supseteq \overline{A_\eta^\alpha} \text{ whenever } \alpha < \text{cf}(\mu) \text{ and } \xi \leq \eta < \mu.$$

Let us put $A_\xi = \bigcup \{A_\xi^\alpha : \alpha < \text{cf}(\mu)\}$ for $\xi < \mu$. The sets A_ξ are pairwise disjoint, so if they are all dense in X then X is μ -resolvable. Thus we can assume that at least one of them is not dense in X , hence there is a nonempty open set $V \subset X$ and an ordinal $\xi^* < \mu$ such that $V \cap A_{\xi^*} = \emptyset$. Then we also have

$$(\ddagger) \quad V \cap A_\eta = \emptyset \text{ for each } \eta \geq \xi^*$$

because of (\dagger) .

For $\kappa < \mu$ pick $\beta < \text{cf}(\mu)$ with $\kappa \leq \mu_\beta$ and put

$$T = \bigcup \{A_\xi^\alpha : \alpha \leq \beta, \xi < \xi^*\}.$$

Then $|T| \leq \mu_\beta \cdot |\xi^*| < \mu$ and it is immediate from our definitions that then we have $d_0(V \setminus T) > \mu_\beta \geq \kappa$. □_{3.12}

Before giving our next result we introduce a refined version of the family of fat sets $\mathcal{H}(Y, \nu)$ defined above and of the associated operator $H(Y, \nu)$. If a cardinal $\varrho < \nu$ is also given, then we let

$$\mathcal{H}(Y, \varrho, \nu) = \{A \in \mathcal{H}(Y, \nu) : \Delta(A) \geq \varrho\}.$$

Again, if $c(X) \leq \nu$ and $\mathcal{H}(Y, \varrho, \nu)$ is non-empty then $\mathcal{H}(Y, \varrho, \nu)$ has a member $H(Y, \varrho, \nu)$ of maximal closure. If $\mathcal{H}(Y, \varrho, \nu)$ is empty then we set $H(Y, \varrho, \nu) = \emptyset$.

Lemma 3.13. *Assume that X is a topological space and μ is a singular cardinal with $c(X) < \mu \leq \Delta(X)$, moreover X satisfies condition $(*)$ from lemma 3.12, i. e. for every $\kappa < \mu$ there is a set $T \in [X]^{<\mu}$ such that $d_0(X \setminus T) > \kappa$. Then we have either (i) or (ii) below.*

- (i) *There is a disjoint family $\{D_\alpha : \alpha < \text{cf}(\mu)\} \subset \mathcal{F}(X) \cap [X]^{<\mu}$ such that $\Delta(D_\alpha)$ converges to μ , moreover*

$$\cup\{D_\gamma : \gamma \geq \alpha\} \in \mathcal{D}(X)$$

for all $\alpha < \text{cf}(\mu)$.

- (ii) *There are an open set $W \in \tau^*(X)$ and a set $T \in [X]^{<\mu}$ with $d_0(W \setminus T) \geq \mu$.*

Proof of 3.13. Fix the same strictly increasing sequence $\langle \mu_\alpha : \alpha < \text{cf}(\mu) \rangle$ of regular cardinals converging to μ with $c(X) \cdot \text{cf}(\mu) < \mu_0$ as in the above proof. Note that then for each $\alpha < \text{cf}(\mu)$ we have

$$\mu_\alpha^- = \sup\{\mu_\beta : \beta < \alpha\} < \mu_\alpha.$$

Then by a straight-forward transfinite recursion on $\alpha < \text{cf}(\mu)$ we define disjoint sets $D_\alpha \in [X]^{<\mu}$ as follows.

If D_β has been defined for each $\beta < \alpha$ then set

$$D_\alpha = H(X \setminus \cup \{D_\beta : \beta < \alpha\}, \mu_\alpha^-, \mu_\alpha).$$

(Note that D_α may be empty but it is a member of $\mathcal{F}(X)$ if it is not.) Next, for each $\alpha < \text{cf}(\mu)$ we let

$$E_\alpha = \cup \{D_\gamma : \gamma \geq \alpha\}.$$

Assume first that $E_\alpha \in \mathcal{D}(X)$ for all $\alpha < \text{cf}(\mu)$. In particular, then $D_\alpha \neq \emptyset$ for cofinally many $\alpha < \text{cf}(\mu)$, hence by re-indexing we may actually assume that $D_\alpha \neq \emptyset$ for all $\alpha < \text{cf}(\mu)$. Now, $\Delta(D_\alpha) > \mu_\alpha^-$ immediately implies that $\Delta(D_\alpha)$ converges to μ , hence (i) is satisfied.

Next, assume that some E_α is not dense, hence there is a $W \in \tau^*(X)$ with $W \cap E_\alpha = \emptyset$. Since X satisfies (*) there is a set $S \in [X]^{<\mu}$ such that $d_0(X \setminus S) > \mu_\alpha$. Let us set

$$T = \cup \{D_\beta : \beta < \alpha\} \cup S,$$

then $|T| < \mu$ as well, moreover we claim that $d_0(W \setminus T) = \kappa \geq \mu$.

Assume, indirectly, that $U \in \tau^*(W)$ and $d(U \setminus T) = \kappa < \mu$. Since $U \setminus T \subset X \setminus S$ we have $\kappa > \mu_\alpha$, hence if $\delta < \text{cf}(\mu)$ is chosen so that

$$\mu_\delta^- \leq \kappa < \mu_\delta$$

then $\alpha < \delta$. Let A be any dense subset of $U \setminus T$ of size κ , then clearly $\Delta(A) = \kappa$ as well, moreover $A \subset X \setminus \cup \{D_\beta : \beta < \delta\}$ holds because $W \cap E_\alpha = \emptyset$. But then,

by our definition, we have

$$A \in \mathcal{H}(X \setminus \cup \{D_\beta : \beta < \delta\}, \mu_\delta^-, \mu_\delta),$$

hence $A \subset \overline{D_\delta}$, contradicting that $W \cap \overline{D_\delta} = \emptyset$. □_{3.13}

We now give one more easy result that, for a limit cardinal λ , may be used to conclude λ -resolvability.

Lemma 3.14. *Let X be a space and λ a limit cardinal and assume that $\{D_\alpha : \alpha < \text{cf}(\lambda)\}$ are disjoint subsets of X such that*

$$\cup \{D_\alpha : \beta \leq \alpha < \text{cf}(\lambda)\} \in \mathcal{D}(X)$$

for every $\beta < \text{cf}(\lambda)$. Assume also that D_α is κ_α -resolvable for each $\alpha < \text{cf}(\lambda)$ and the sequence $\langle \kappa_\alpha : \alpha < \text{cf}(\lambda) \rangle$ converges to λ . Then X is λ -resolvable.

Proof of 3.14. For each $\alpha < \text{cf}(\lambda)$ fix a disjoint family

$$\{E_\xi^\alpha : \xi < \kappa_\alpha\} \subset \mathcal{D}(D_\alpha),$$

then for any $\xi < \lambda$ set

$$E_\xi = \cup \{E_\xi^\alpha : \xi < \kappa_\alpha\}.$$

Since the κ_α converge to λ , for any fixed $\xi < \lambda$ we eventually have $\xi < \kappa_\alpha$ and so E_ξ is dense in X . Consequently the disjoint family $\{E_\xi : \xi < \lambda\}$ witnesses that X is λ -resolvable. □

From the above results and the first part of theorem 3.6 we may now easily obtain the "missing" second part. Indeed, assume that λ is singular and $s(X) <$

$\Delta(X) = \lambda$. Reasoning inductively, we may assume that if $s(Y) < \Delta(Y) < \lambda$ then Y is maximally, that is $\Delta(Y)$ -resolvable.

Now, by lemma 4.1, to prove that X is λ -resolvable it suffices to show that some subspace of X is. Since $c(X) \leq s(X)$, from lemmas 3.12 and 3.13 it follows that, if X itself is not λ -resolvable, then either there are a $W \in \tau^*(X)$ and a $T \in [W]^{<\lambda}$ such that $d_0(W \setminus T) \geq \lambda$ or there is a $V \in \tau^*(X)$ with disjoint sets $\{D_\alpha : \alpha < \text{cf}(\lambda)\} \subset \mathcal{F}(V)$ such that $\Delta(D_\alpha)$ converges to λ and

$$\cup\{D_\gamma : \alpha \leq \gamma < \text{cf}(\lambda)\} \in \mathcal{D}(X)$$

for all $\alpha < \text{cf}(\lambda)$. But we have seen that in the first case $W \setminus T$ (and hence W), while in the second V is λ -resolvable.

We are now ready to present our result that, under certain conditions, enables us to deduce λ -resolvability from $(< \lambda)$ -resolvability for a singular cardinal λ . We first recall that $\widehat{c}(X)$ is defined as the smallest (uncountable) cardinal such that X has no disjoint family of open sets of that size. As was shown in [14] (see also [22, 4.1]), $\widehat{c}(X)$ is always a regular cardinal. We also note that if λ is a limit cardinal then every $(< \lambda)$ -resolvable space S has dispersion character $\Delta(S) \geq \lambda$.

Theorem 3.15. *Assume that X is a topological space, λ is a singular cardinal, and $\widehat{c}(X) \leq \text{cf}(\lambda) < \lambda \leq \Delta(X)$. If every dense subspace $S \subset X$ satisfying $\Delta(S) \geq \lambda$ is $(< \lambda)$ -resolvable then X is actually λ -resolvable.*

Proof of 3.15. Let us start by pointing out that if A is fat in X then $S = A \cup (X \setminus \overline{A}) \in \mathcal{D}(X)$, moreover $\Delta(A) \geq \lambda$ implies $\Delta(S) \geq \lambda$. So, every fat set

$A \in \mathcal{F}(X)$ that satisfies $\Delta(A) \geq \lambda$ is $(< \lambda)$ -resolvable. It immediately follows from this that the conditions on our space X are inherited by all non-empty open subspaces, hence by lemma 4.1 it is again sufficient to prove that X has some λ -resolvable subspace.

Now, if some $A \in \mathcal{F}(X)$ satisfies $d_0(A) \geq \lambda$ then $\Delta(B) \geq \lambda$ holds for every $B \in \mathcal{D}(A)$, hence all dense subsets of A are $(< \lambda)$ -resolvable. But then, by lemma 3.11, A is λ -resolvable.

Therefore, from here on we may assume that $d_0(A) < \lambda$ for all $A \in \mathcal{F}(X)$. Actually, we claim that then even $d(A) < \lambda$ holds whenever $A \in \mathcal{F}(X)$. Indeed, if $A \in \mathcal{D}(U)$ for some $U \in \tau^*(X)$ then let \mathcal{W} be a maximal disjoint family of open sets $W \subset U$ such that $d(A \cap W) < \lambda$. Then $\widehat{c}(X) \leq \text{cf}(\lambda) = \kappa$ implies $|\mathcal{W}| < \kappa$, moreover $\cup \mathcal{W}$ is clearly dense in U by our assumption. But then $\cup \mathcal{W} \cap A$ is dense in A and so

$$d(A) \leq d(\cup \mathcal{W} \cap A) = \sum \{d(W \cap A) : W \in \mathcal{W}\} < \lambda.$$

(We note that this is the only part of the proof where $\widehat{c}(X) \leq \text{cf}(\lambda)$ is used rather than the weaker assumption $c(X) < \lambda$.)

By lemma 3.12, if X itself is not λ -resolvable then there is a $V \in \tau^*(X)$ that satisfies condition (*). We shall show that then V is λ -resolvable.

To see this, first fix a strictly increasing sequence $\langle \lambda_\alpha : \alpha < \kappa \rangle$ of cardinals converging to λ and then, using (*), fix for each $\alpha < \kappa$ a set $T_\alpha \in [V]^{< \lambda}$ with $d_0(V \setminus T_\alpha) > \lambda_\alpha$. Having done this, we define disjoint sets $D_\alpha \in \mathcal{D}(V) \cap [V]^{< \lambda}$ by transfinite induction on $\alpha < \kappa$ as follows.

Assume that $\alpha < \kappa$ and $D_\beta \in \mathcal{D}(V) \cap [V]^{<\lambda}$ has been defined for each $\beta < \alpha$.

Set

$$Z_\alpha = X \setminus (\cup\{D_\beta : \beta < \alpha\} \cup T_\alpha),$$

then Z_α is dense in V because $\Delta(X) \geq \lambda$. But then $d(Z_\alpha) < \lambda$, hence we may pick $D_\alpha \in \mathcal{D}(Z_\alpha) \subset \mathcal{D}(V)$ with $|D_\alpha| < \lambda$. Note that as $D_\alpha \subset V \setminus T_\alpha$ we also have $\Delta(D_\alpha) > \lambda_\alpha$.

Now consider any partition $\{J_\xi : \xi < \kappa\}$ of κ into κ many sets of size κ and for each $\xi < \kappa$ put

$$E_\xi = \cup\{D_\alpha : \alpha \in J_\xi\}.$$

Then each E_ξ is dense in V and clearly $\Delta(E_\xi) = \lambda$, hence it is $(< \lambda)$ -resolvable. But the E_ξ 's are pairwise disjoint, hence obviously V is λ -resolvable. $\square_{3.15}$

We do not know if the assumption $\widehat{c}(X) \leq \text{cf}(\lambda)$ can be relaxed to $c(X) < \lambda$ in theorem 3.15, or even if it can be dropped altogether.

3.2 A simpler proof of Pavlov's theorem concerning extent

The *extent* $e(X)$ of a space X is defined as the supremum of sizes of all closed discrete subspaces of X . (This is Archangelskii's notation, in [31] $\text{ext}(X)$ and in [22] $p(X)$ is used to denote the same cardinal function.) Similarly as in the previous section for the spread $s(X)$, we may define $\widehat{e}(X)$ as the smallest infinite (but not necessarily uncountable) cardinal such that X has no closed discrete subset of that size. Note that a space X is countably compact iff $\widehat{e}(X) = \omega$. Clearly, one has $\widehat{e}(X) \leq \text{pe}(X)$ (the latter was defined in the abstract).

In [31] it was proved that $\Delta(X) > \text{pe}(X)$ implies the ω -resolvability of X for any T_3 space X . In this section we shall present our proof of the slightly stronger result in which only $\Delta(X) > \widehat{\text{e}}(X)$ is used. We believe that this proof is significantly simpler than the one given in [31], although it follows the same steps.

We start with giving our simplified proof of the following result of Pavlov concerning spaces that are finite unions of left separated subspaces.

Theorem 3.16. (PAVLOV)[31, LEMMA 3.1] *Assume that $\text{ls}(X) < \omega$ and $\kappa \leq |X|$ is an uncountable regular cardinal. Then there is a strictly increasing and continuous sequence $\langle F_\alpha : \alpha < \kappa \rangle$ of closed subsets of X with $|F_\alpha| < \kappa$ for all $\alpha < \kappa$.*

Proof. We prove the theorem by induction on $\text{ls}(X)$. So assume that it is true for $\text{ls}(X) = k$ and consider $X = \bigcup_{0 \leq i \leq k} L_i$ where the L_i are disjoint and left separated, moreover $\omega < \kappa \leq |X|$. We may clearly assume that the left separating order type of each L_i is $\leq \kappa$.

Assume that S is an initial segment of some L_i with $\text{tp}(S) < \kappa$ and $|\overline{S}| \geq \kappa$ (closures are always taken in X). Since $\overline{S} \cap L_i = S$ we may apply the inductive hypothesis to $\overline{S} \setminus S$ and find an increasing and continuous κ -sequence $\langle F_\alpha : \alpha < \kappa \rangle$ of its closed subsets of size $< \kappa$. But then the traces $\overline{F_\alpha} \cap S$ will stabilize and $|\overline{F_\alpha}| \leq |F_\alpha| + |S| < \kappa$, hence a suitable final segment of $\langle \overline{F_\alpha} : \alpha < \kappa \rangle$ is as required. Almost the same argument shows that the inductive step can also be completed if $|L_i| < \kappa$ for some i . So we may assume that $\text{tp } L_i = \kappa$ for each i

and that $|\overline{A}| < \kappa$ whenever $A \in [X]^{<\kappa}$.

Let y_α denote the α th member of L_0 and use the inductive assumption to find an increasing and continuous κ -sequence $\langle F_\alpha : \alpha < \kappa \rangle$ of closed subsets of $\bigcup_{1 \leq i \leq k} L_i$ of size $< \kappa$, and then consider the set

$$I = \{\alpha < \kappa : y_\alpha \in \overline{F_\alpha}\}.$$

Assume first that $|I| < \kappa$ and hence $\sigma = \sup I < \kappa$. We claim that then the set

$$J = \{\beta > \sigma : \overline{F_\beta} \neq \bigcup_{\gamma < \beta} \overline{F_\gamma}\}$$

is non-stationary in κ . Indeed, for each $\beta \in J$ there must be some $g(\beta) < \kappa$ with $y_{g(\beta)} \in \overline{F_\beta} \setminus \bigcup_{\gamma < \beta} \overline{F_\gamma}$. Since $g(\beta) \geq \beta > \sigma$ would imply $g(\beta) \notin I$ and hence

$$y_{g(\beta)} \notin \overline{F_{g(\beta)}} \supset \overline{F_\beta},$$

we must have $g(\beta) < \beta$. But the regressive function g is clearly one-to-one on J , hence by Fodor's (or Neumer's) pressing down theorem J is non-stationary. So there is a club set C in κ with $C \cap J = \emptyset$, and then the sequence $\langle \overline{F_\alpha} : \alpha \in C \setminus \sigma \rangle$ clearly satisfies our requirements.

So we may assume that $|I| = \kappa$. For each $\alpha < \kappa$ let us put $H_\alpha = \overline{\{y_\gamma : \gamma \in I \cap \alpha\}}$. Note that we have $H_\alpha \subset \overline{F_\alpha}$ by the definition of I . Next, consider the set

$$J = \{\alpha < \kappa : \alpha \text{ is limit and } H_\alpha \neq \bigcup_{\gamma < \alpha} H_\gamma\}.$$

We claim that this set J is again non-stationary. Indeed, for every $\alpha \in J$ we may pick a "witness" $z_\alpha \in H_\alpha \setminus \bigcup_{\gamma < \alpha} H_\gamma$. Now, if $z_\alpha \in L_0$ then $z_\alpha = y_{g(\alpha)}$ for

some $g(\alpha) < \alpha$ because L_0 is left separated. If, on the other hand, $z_\alpha \notin L_0$ then $z_\alpha \in H_\alpha \subset \overline{F_\alpha}$ implies $z_\alpha \in F_\alpha$ because F_α is closed in $X \setminus L_0$. But the sequence $\langle F_\alpha : \alpha \in \kappa \rangle$ is continuous, hence in this case we can choose an ordinal $g(\alpha) < \alpha$ such that $z_\alpha \in F_{g(\alpha)}$.

In other words, this means that if $g(\alpha) = \beta$ then $z_\alpha \in \{y_\beta\} \cup F_\beta$. Now, the sequence $\langle z_\alpha : \alpha \in J \rangle$ is obviously one-to-one, hence for each $\beta < \kappa$ we have $|g^{-1}\{\beta\}| \leq |F_\beta| + 1 < \kappa$, consequently, again by Fodor, J is not stationary. So there is a club $C \subset \kappa \setminus J$ and then $\langle H_\alpha : \alpha \in C \rangle$ is increasing and continuous, however maybe it is not strictly increasing. But $|I| = \kappa$ clearly implies that the union of the H_α 's is of size κ and so an appropriate subsequence of $\langle H_\alpha : \alpha \in C \rangle$ will be both continuous and strictly increasing. \square

Before proceeding further, we need a simple definition.

Definition 3.17. Let X be a space and μ an infinite cardinal number. We say that $x \in X$ is a T_μ point of X if for every set $A \in [X]^{<\mu}$ there is some $B \in [X \setminus A]^{<\mu}$ such that $x \in \overline{B}$. We shall use $T_\mu(X)$ to denote the set of all T_μ points of X .

For the reader familiar with Pavlov's paper [31] we note that his $\text{tr}_{\nu^+, \nu}(X)$ is identical with our $T_{\nu^+}(X)$. Note also that if $Y \subset X$ then trivially any T_μ point in Y is a T_μ point in X , that is, we have $T_\mu(Y) \subset T_\mu(X)$. Finally, if μ is regular then the set $T_\mu(X)$ is clearly $(< \mu)$ -closed in X , i. e. for every set $A \in [T_\mu(X)]^{<\mu}$ we have $\overline{A} \subset T_\mu(X)$.

Lemma 3.18. *Assume that the space X may be written as the union of a strictly increasing continuous chain $\langle F_\alpha : \alpha < \kappa \rangle$ of closed subsets of X with $|F_\alpha| < \kappa$ for*

all $\alpha < \kappa$, where κ is an uncountable regular cardinal. Then $T_\kappa(X) = \emptyset$ implies that there exists a set $D \subset X$ with $|D| = \kappa$ such that every subset $Y \in [D]^{<\kappa}$ is closed discrete in X . In particular, we have $\widehat{e}(X) \geq \kappa$.

Proof. The assumption $T_\kappa(X) = \emptyset$ implies that for every point $x \in X$ we may fix a set $A_x \in [X]^{<\kappa}$ such that $x \notin \overline{B}$ whenever $B \in [X \setminus A_x]^{<\kappa}$. By the regularity of κ , the set

$$C = \{\alpha < \kappa : \forall x \in F_\alpha (A_x \subset F_\alpha)\}$$

is club in κ . For each $\alpha \in C$ let us pick a point $x_\alpha \in F_{\alpha+1} \setminus F_\alpha$ and then set $D = \{x_\alpha : \alpha \in C\}$.

To see that this D is as required, it remains to show that all "small" subsets of D are closed discrete. This in turn will follow if we show that all proper initial segments of D are. So let $\gamma < \kappa$ and consider the set $S = \{x_\alpha : \alpha \in C \cap \gamma\}$. For every point $y \in X$ there is a $\beta < \kappa$ such that $y \in F_{\beta+1} \setminus F_\beta$. Let δ be the largest element of C with $\delta \leq \beta$ and ε the smallest element of C above β , hence we have $\delta \leq \beta < \varepsilon$.

Then, on one hand, $\{x_\alpha : \alpha < \delta\} \subset F_\delta \subset F_\beta$, while on the other hand $A_y \subset F_\varepsilon$ and $\{x_\alpha : \varepsilon \leq \alpha < \gamma\} \subset X \setminus F_\varepsilon$, which together imply that y has a neighbourhood U such that $U \cap S \subset \{x_\delta\}$. \square

We need one more result making use of the operator T_μ .

Lemma 3.19. *If a space X satisfies $T_\mu(X) = X$ for a regular cardinal μ then X is μ -resolvable.*

Proof. Clearly, $T_\mu(X) = X$ implies $T_\mu(U) = U$ for all open subsets $U \subset X$, hence by lemma 4.1 it suffices to show that X includes a μ -resolvable subspace Y .

Since every point of X is a T_μ point, for any set $A \in [X]^{<\mu}$ we may fix a disjoint family $\mathcal{B}(A) \subset [X \setminus A]^{<\mu}$ with $|\mathcal{B}(A)| = |A| < \mu$ such that

$$A \subset \cup \{\overline{B} : B \in \mathcal{B}(A)\}.$$

We now define sets A_α in $[X]^{<\mu}$ by induction on $\alpha < \mu$ as follows. Let $x \in X$ be any point and start with $A_0 = \{x\}$. Assume next that $0 < \alpha < \mu$ and the sets $A_\beta \in [X]^{<\mu}$ have been defined for all $\beta < \alpha$. Then we set

$$B_\alpha = \bigcup \mathcal{B}(\cup \{A_\beta : \beta < \alpha\}) \text{ and } A_\alpha = \cup \{A_\beta : \beta < \alpha\} \cup B_\alpha.$$

After the induction is completed we let

$$Y = \cup \{A_\alpha : \alpha < \mu\}.$$

It is clear from the construction that the B_α 's are pairwise disjoint, moreover for every set $s \in [\mu]^\mu$ the union $\cup_{\alpha \in s} B_\alpha$ is dense in Y . But then Y is obviously μ -resolvable.

□

We are now ready to state and prove our promised result.

Theorem 3.20. *Assume that the regular closed subsets of the space X form a π -network in X and $T_\mu(X)$ is dense in X for some regular cardinal $\mu > \widehat{e}(X)$. Then X is ω -resolvable. In particular, any T_3 space X satisfying $\Delta(X) > \widehat{e}(X)$ is ω -resolvable.*

Proof. Assume, indirectly, that X is ω -irresolvable. By lemmas 4.1 and 3.2 then there is a regular closed subset K of X that is both *hereditarily* ω -irresolvable and satisfies $\text{ls}(K) < \omega$.

Let us now define the sequence of sets $\{K_n : n < \omega\}$ by the following recursion: $K_0 = K$ and $K_{n+1} = T_\mu(K_n)$. Since $T_\mu(Y)$ is $(< \mu)$ -closed in Y for any space Y , we may conclude by a simple induction that K_i is $(< \mu)$ -closed in K and hence $\widehat{e}(K_i) \leq \widehat{e}(K) < \mu$ for all $i < \omega$.

We next claim that, for each $n < \omega$, $K_{n+1} = T_\mu(K_n)$ is dense in K_n and hence in K . For $n = 0$ this follows immediately from our assumption that $T_\mu(X) \in \mathcal{D}(X)$.

Clearly, any neighborhood of a T_μ point in any space must have size at least μ . Hence if our claim holds up to (and including) n then we also have $\Delta(K_n) \geq \mu$ and since $K_n \in \mathcal{D}(K)$ the regular closed subsets of K_n form a π -network in K_n . (The latter holds because the regular closed subsets of a dense subspace are exactly the traces of the regular closed sets in the original space.)

Now, let U be any non-empty open subset of K_n . We show first that then $|U \cap K_{n+1}| \geq \mu$, hence $\Delta(K_{n+1}) \geq \mu$. (In other words, K_{n+1} is not only dense but even μ -dense in K_n .) To see this, let $\emptyset \neq F \subset U$ be regular closed in K_n , then $|F| \geq \mu$ and $\text{ls}(F) < \omega$ imply, in view of theorem 3.16, the existence of a strictly increasing continuous sequence $\langle F_\alpha : \alpha < \mu \rangle$ of closed subsets of F (and hence of X) with $|F_\alpha| < \mu$. Then we may apply lemma 3.18 to any final segment of the sequence $\langle F_\alpha : \alpha < \mu \rangle$ to conclude that $F_\alpha \cap T_\mu(K_n) = F_\alpha \cap K_{n+1} \neq \emptyset$ for cofinally many $\alpha < \mu$, hence $|U \cap K_{n+1}| \geq |F \cap K_{n+1}| \geq \mu$.

But $\Delta(K_{n+1}) \geq \mu$ implies that for any non-empty regular closed set H in K_{n+1} we have $|H| \geq \mu$, and so, using again $\text{ls}(H) < \omega$ and $\widehat{\text{e}}(K_n) < \mu$, we obtain from theorem 3.16 and lemma 3.18 that $T_\mu(H)$ is non-empty, i. e. K_{n+2} is indeed dense in K_{n+1} .

Now suppose that there is an $n < \omega$ such that $K_n \setminus K_{n+1}$ is not dense in K_n . This would imply that for some $U \in \tau^*(K_n)$ we have $U \subset K_{n+1}$ and hence $T_\mu(U) = U$. But that would imply by lemma 3.19 that U is μ -resolvable, a contradiction. Therefore, we must have that $K_n \setminus K_{n+1}$ is dense in K_n and hence in K for all $n < \omega$. But then K would be ω -resolvable, which is again absurd. This contradiction then completes the proof of the first part of our theorem.

To see the second part note that, by lemma 3.2 and by considering regular closed subsets of X , it suffices to prove the ω -resolvability of X under the additional condition $\text{ls}(X) < \omega$. But then $T_\mu(X) \in \mathcal{D}(X)$ follows immediately from theorem 3.16 and lemma 3.18 with the choice $\mu = \widehat{\text{e}}(X)^+$. \square

Since for any crowded (i. e. dense-in-itself) countably compact T_3 space X one has $\Delta(X) \geq \mathfrak{c} \geq \omega_1$, theorem 3.20 immediately implies the following result of Comfort and Garcia-Ferreira.

Theorem (Comfort, Garcia-Ferreira, [7, Theorem 6.9]). *Every crowded and countably compact T_3 space is ω -resolvable.*

Note that the assumption of regularity in this theorem is essential because of the following two results.

Theorem (Malykhin, [27, Example 14]). *There is a countably compact, irresolvable T_2 space.*

Theorem (Pavlov, [31, Example 3.9]). *There is a countably compact, irresolvable Uryshon space.*

Pytkeev has recently announced in [33] that a crowded and countably compact T_3 space is even ω_1 -resolvable. We haven't seen his paper but would like to point out that this stronger result is an immediate consequence of an old (and deep) result of Tkačenko and of lemma 3.2.

Tkačenko proved in [36] that *if X is a countably compact T_3 space with $\text{ls}(X) \leq \omega$ then X is compact and scattered.* In [16] it was shown that this statement remains valid if T_3 is weakened to T_2 , hence we get the following result.

Theorem 3.21. *If X is a crowded and countably compact T_2 space in which the regular closed subsets form a π -network then X is ω_1 -resolvable.*

Proof of theorem 3.21. By the above result from [16], every non-empty regular closed subset $F \subset X$ must satisfy $\text{ls}(F) \geq \omega_1$. But then X is ω_1 -resolvable by lemma 4.1. □_{3.21}

Any crowded and countably compact T_3 space has dispersion character $\geq \mathfrak{c}$. Hence the following interesting, and apparently difficult, problem is left open by theorem 3.21.

Problem 3.22. *Is a crowded and countably compact T_3 space \mathfrak{c} -resolvable or even maximally resolvable?*

4 Monotonically normal spaces

4.1 Countable resolvability

For a topological space X we denote by $\mathcal{D}(X)$ the family of all dense subsets of X and by $\mathcal{N}(X)$ the ideal of all nowhere dense sets in X . Given a cardinal $\kappa > 1$, the space X is called κ -*resolvable* iff it contains κ many disjoint dense subsets. We say that X is *almost κ -resolvable* if there are κ many dense sets whose pairwise intersections are nowhere dense, that is we have $\{D_\alpha : \alpha < \kappa\} \subset \mathcal{D}(X)$ such that $D_\alpha \cap D_\beta \in \mathcal{N}(X)$ if $\alpha \neq \beta$. X is maximally resolvable iff it is $\Delta(X)$ -resolvable, where $\Delta(X) = \min\{|G| : G \neq \emptyset \text{ open}\}$ is called the dispersion character of X . Finally, if X is *not* κ -resolvable then it is also called κ -*irresolvable*.

The following simple but useful fact, in the case of κ -resolvability, was observed by El'kin in [12].

Lemma 4.1. *A space X is κ -resolvable (almost κ -resolvable) iff every nonempty open set in X includes a nonempty (and open) κ -resolvable (almost κ -resolvable) subset.*

The aim of this section is to investigate the (almost) resolvability properties of monotonically normal spaces. Since the most important examples of monotonically normal spaces are metric and linearly ordered spaces that are all known to be maximally resolvable, this aim seems to be both natural and justified to us. We hope that our results, by turning out to be both surprising and non-trivial, will also convince the reader about this.

Let us next recall the definition of monotonically normal spaces. For any topological space X we write

$$\mathcal{M}(X) = \{\langle x, U \rangle \in X \times \tau(X) : x \in U\}.$$

The elements of $\mathcal{M}(X)$ will be referred to as *marked open sets*. The space X is called *monotonically normal* iff it is T_1 and it admits a *monotone normality operator*, that is a function $H : \mathcal{M}(X) \longrightarrow \tau(X)$ such that

- (1) $x \in H(x, U) \subset U$ for each $\langle x, U \rangle \in \mathcal{M}(X)$,
- (2) if $H(x, U) \cap H(y, V) \neq \emptyset$ then $x \in V$ or $y \in U$.

We call a set D in a space X *strongly discrete* if the points in D may be separated by pairwise disjoint neighborhoods. It is well-known that in a monotonically normal space any discrete subset is strongly discrete. On the other hand, in [10] it was proved that every non-isolated point of a monotonically normal space is the accumulation point of a discrete subspace. Consequently, one obtains the following result.

Theorem 4.2 ([10]). *In a monotonically normal space any non-isolated point is the accumulation point of some strongly discrete set.*

Let us say that a space X is SD if it has the property described in theorem 4.2, that is every non-isolated point of X is the accumulation point of some strongly discrete set.

Theorem 4.3. *Any crowded SD space X is ω -resolvable.*

Proof. The SD property is clearly hereditary for open subspaces. Hence, by lemma 4.1, it suffices to prove that X includes an ω -resolvable subspace.

First we show that for every strongly discrete $D \subset X$ there is a strongly discrete $E \subset X \setminus \overline{D}$ such that $D \subset \overline{E}$. Indeed, fix a neighbourhood assignment U_d on D that separates D and for each $d \in D$ pick a strongly discrete set $E_d \subset X \setminus \{d\}$ with $d \in \overline{E_d}$. Then $E = \bigcup_{d \in D} (E_d \cap U_d)$ is clearly as claimed.

Now pick an arbitrary point $x \in X$ and set $E_0 = \{x\}$. Using the above claim, for each $n < \omega$ we can inductively define a strongly discrete set $E_{n+1} \subset X \setminus \overline{E_n}$ such that $E_n \subset \overline{E_{n+1}}$. Since $\bigcup_{i \leq n} E_i \subset \overline{E_n}$, the sets $\{E_n : n < \omega\}$ are pairwise disjoint. Let us finally set $E = \bigcup \{E_n : n < \omega\}$. It is clear from our construction that if $I \subset \omega$ is infinite then $\bigcup \{E_n : n \in I\}$ is dense in E , so the subspace E of X is obviously ω -resolvable. \square

Corollary 4.4. *Every crowded monotonically normal space is ω -resolvable.*

4.2 H-sequences and almost resolvability

The main result of the previous section, namely that (crowded) monotonically normal spaces are ω -resolvable, used very little of the particular structure provided by monotone normality. In this section we shall describe a procedure on monotonically normal spaces that is quite specific in this respect and so, not surprisingly, it leads to some stronger (almost) resolvability results. This procedure had been originated (in a different form) by S. Williams and H. Zhou in [37].

Definition 4.5. Let H be a monotone normality operator on a space X . A family

$\mathcal{E} \subset \mathcal{M}(X)$ of marked open sets is said to be *H-disjoint* if for any two members $\langle x, U \rangle, \langle y, V \rangle$ of \mathcal{E} we have $H(x, U) \cap H(y, V) = \emptyset$. Clearly, if \mathcal{E} is H-disjoint then $D(\mathcal{E}) = \{x : \exists U \text{ with } \langle x, U \rangle \in \mathcal{E}\}$ is (strongly) discrete.

By Zorn's lemma, for every open set G in X we can fix a *maximal* H-disjoint family $\mathcal{E}(G) \subset \mathcal{M}(G)$ with the additional property that $\overline{U} \subset G$ whenever $\langle x, U \rangle \in \mathcal{E}(G)$. The maximality of $\mathcal{E}(G)$ implies that

$$\bigcup \{H(x, U) : \langle x, U \rangle \in \mathcal{E}(G)\}$$

is a dense open subset of G .

With the help of the above defined operator $\mathcal{E}(G)$ we may now describe our basic procedure as follows.

Definition 4.6. A sequence $\langle \mathcal{E}_\alpha : \alpha < \delta \rangle$ is called a completed H-sequence of X iff

1. $\mathcal{E}_0 = \mathcal{E}(X)$,
2. for each $\alpha < \delta$ we have

$$\mathcal{E}_{\alpha+1} = \bigcup \{ \mathcal{E}(H(x, U) \setminus \{x\}) : \langle x, U \rangle \in \mathcal{E}_\alpha \},$$

3. if $\alpha < \delta$ is a limit ordinal then the family

$$\mathcal{W}_\alpha = \{W \in \tau(X) : \forall \beta < \alpha \exists \langle x, U \rangle \in \mathcal{E}_\beta \text{ with } W \subset U\}$$

is a π -base in X (or, equivalently, its union $\cup \mathcal{W}_\alpha$ is dense in X) and \mathcal{E}_α is a maximal H-disjoint collection of marked open sets $\langle y, V \rangle$ with $V \in \mathcal{W}_\alpha$,

4. the family

$$\mathcal{W}_\delta = \{W \in \tau(X) : \forall \beta < \delta \exists \langle x, U \rangle \in \mathcal{E}_\beta \text{ with } W \subset U\}$$

is *not* a π -base in X .

The reader may convince himself by a straight-forward transfinite induction that the following fact is valid.

Fact 4.7. *Every crowded monotonically normal space X , with monotone normality operator H , admits a completed H -sequence $\langle \mathcal{E}_\alpha : \alpha < \delta \rangle$ where δ is necessarily a limit ordinal.*

We now fix some notation concerning a given completed H -sequence $\langle \mathcal{E}_\alpha : \alpha < \delta \rangle$ of X . For any ordinal $\alpha < \delta$ we put $D_\alpha = D(\mathcal{E}_\alpha)$ and $H_\alpha = \bigcup \{H(x, U) : \langle x, U \rangle \in \mathcal{E}_\alpha\}$. It is clear from our definitions that each H_α is dense open in X , moreover $\beta < \alpha < \delta$ implies that $H_\beta \supset H_\alpha$ and $D_\beta \cap H_\alpha = \emptyset$. If $I \subset \delta$ is a set of ordinals we write $D[I] = \bigcup \{D_\alpha : \alpha \in I\}$. Finally, we set $V = X \setminus \overline{\bigcup \mathcal{W}_\delta}$, then V is a non-empty open set in X .

Lemma 4.8. *If I is bounded in δ then $D[I]$ is nowhere dense in X . However, if I is unbounded in δ then $D[I]$ is dense in V , that is we have $V \subset \overline{D[I]}$.*

Proof. The first part is obvious because $I \subset \alpha < \delta$ implies $D[I] \cap H_\alpha = \emptyset$.

Assume now that I is cofinal in δ but, arguing indirectly, for some $G \in \tau^*(V)$ we have $G \cap D[I] = \emptyset$. Pick any point $z \in G$, we claim that then, for all $\alpha < \delta$ and $\langle x, U \rangle \in \mathcal{E}_\alpha$, $H(x, U) \cap H(z, G) \neq \emptyset$ implies $z \in H(x, U)$.

Indeed, if $\beta \in (\alpha, \delta) \cap I$ then there is $\langle x', U' \rangle \in \mathcal{E}_\beta$ with

$$H(x', U') \cap H(x, U) \cap H(z, G) \neq \emptyset$$

because H_β is dense in X . It follows that $U' \subset H(x, U)$, hence $x' \notin G$ as $x' \in D_\beta$ and $G \cap D_\beta = \emptyset$. But then $H(x', U') \cap H(z, G) \neq \emptyset$ implies $z \in U' \subset H(x, U)$.

The sets $\{H(x, U) : \langle x, U \rangle \in \mathcal{E}_\alpha\}$ being pairwise disjoint, it follows that for each $\alpha < \delta$ there is exactly one $\langle x_\alpha, U_\alpha \rangle \in \mathcal{E}_\alpha$ such that $H(x_\alpha, U_\alpha) \cap H(z, G) \neq \emptyset$. But then $H(z, G) \subset \overline{H(x_\alpha, U_\alpha)} \subset \overline{U_\alpha}$ whenever $\alpha < \delta$, consequently

$$H(z, G) \subset \overline{U_{\alpha+1}} \subset U_\alpha$$

as well. This, however, would imply $H(z, G) \in \mathcal{W}_\delta$, contradicting that $H(z, G) \subset G \subset V$. \square

We may now give the main result of this section.

Theorem 4.9. *Any crowded monotonically normal space X is almost $\min(\mathfrak{c}, \omega_2)$ -resolvable. So X is almost ω_1 -resolvable, and even almost ω_2 -resolvable if the continuum hypothesis (CH) fails.*

Proof. By lemma 4.1 it suffices to show that some non-empty open $V \subset X$ satisfies this property. To see this, let us consider a completed H-sequence $\langle \mathcal{E}_\alpha : \alpha < \delta \rangle$ of X . Let I be a cofinal subset of δ of order type $\text{cf}(\delta)$ and $\{I_\zeta : \zeta < \mu\}$ be an almost disjoint subfamily of $[I]^{\text{cf}(\delta)}$, where $\mu = 2^\omega = \mathfrak{c}$ if $\text{cf}(\delta) = \omega$ and $\mu = \text{cf}(\delta)^+ \geq \omega_2$ if $\text{cf}(\delta) > \omega$. Then the family $\{D[I_\zeta] : \zeta < \mu\}$ witnesses that V is almost μ -resolvable. \square

Since almost ω -resolvability is clearly equivalent to ω -resolvability, theorem 4.9 provides us a new proof of 4.4.

4.3 Spaces from trees and ultrafilters

Since the prime examples of monotonically normal spaces, namely metric and ordered spaces, are all maximally resolvable, the results of the two preceding sections seem rather modest. The main aim of this section is to show that, at least modulo some large cardinals, nothing stronger than ω -resolvability can be expected of a monotonically normal space X , even if its dispersion character $\Delta(X)$ is large. The examples that show this have actually been around but, as far as we know, the fact that they are monotonically normal has not been noticed.

The underlying set of such a space is an *everywhere infinitely branching tree* $\langle T, < \rangle$. This simply means that for each $t \in T$ the set S_t of all immediate successors of t in T is infinite. The height of such a tree is obviously a limit ordinal. (In fact, nothing is lost if we only consider trees of height ω .) By a *filtration* on T we mean a map F with domain T that assigns to every $t \in T$ a filter $F(t)$ on S_t such that every cofinite subset of S_t belongs to $F(t)$ (that is, $F(t)$ extends the Fréchet filter on S_t).

Definition 4.10. Assume that F is a filtration on an everywhere infinitely branching tree $\langle T, < \rangle$. A topology τ_F is then defined on T by

$$\tau_F = \{V \subset T : \forall t \in V (V \cap S_t \in F(t))\},$$

and the space $\langle T, \tau_F \rangle$ is denoted by $X(F)$.

Theorem 4.11. *Let F be a filtration on an everywhere infinitely branching tree $\langle T, < \rangle$. Then the space $X(F)$ is monotonically normal.*

Proof. That τ_F is indeed a topology that satisfies the T_1 separation axiom is obvious and well-known. The novelty is in showing that $X(F)$ is monotonically normal.

To this end we define $H(s, V)$ for $s \in V \in \tau_F$ as follows:

$$H(s, V) = \{t \in V : s \leq t \text{ and } [s, t] \subset V\}.$$

Of course, here $[s, t] = \{r : s \leq r \leq t\}$. Clearly, $H(s, V) \in \tau_F$ and $s \in H(s, V) \subset V$.

Next, assume that $t \in H(s_1, V_1) \cap H(s_2, V_2)$. Then $s_1, s_2 \leq t$ implies that s_1 and s_2 are comparable, say $s_1 \leq s_2$. But then we have $s_2 \in [s_1, t] \subset V_1$, consequently H is indeed a monotone normality operator on $X(F)$. \square

Of special interest are those filtrations F for which $F(t)$ is a (free) ultrafilter on S_t for all $t \in T$. Such an F will be called an *ultrafiltration*. In this case we have a convenient way to determine the closures of sets in the space $X(F)$ that will be put to good use later.

Definition 4.12. For every set $A \subset T$ we define

$$C(A) = A \cup \{t \in T : S_t \cap A \in F(t)\}.$$

Then by transfinite recursion we define $C^\alpha(A)$ for all ordinals α by $C^{\alpha+1}(A) = C(C^\alpha(A))$ for successors and $C^\alpha(A) = \cup\{C^\beta : \beta < \alpha\}$ for α limit.

Lemma 4.13. *Let F be an ultrafiltration on the tree T . Then a set $B \subset T$ is closed in $X(F)$ iff $B = C(B)$. Consequently, for any subset $A \subset T$ there is an ordinal $\alpha < |T|^+$ with $\overline{A} = C^\alpha(A)$.*

Proof. First, if $B = C(B)$ then for each $t \in T \setminus B$ we have $S_t \cap B \notin F(t)$, hence $S_t \setminus B \in F(t)$ because $F(t)$ is an ultrafilter. Then $T \setminus B$ is open by the definition of τ_F , hence B is closed. Conversely, if B is closed in $X(F)$ then for each $t \in T \setminus B$ we have $S_t \setminus B \in F(t)$, hence $S_t \cap B \notin F(t)$, that is $t \notin C(B)$. But this means that $B = C(B)$.

Next, $C(A) \subset \overline{A}$ is obvious, and then by induction we get $C^\alpha(A) \subset \overline{A}$ for all α . But for some $\alpha < |T|^+$ we must have $C(C^\alpha(A)) = C^\alpha(A)$, and then $\overline{A} = C^\alpha(A)$ for $C^\alpha(A)$ is closed by the above. \square

Let \mathfrak{u} be an ultrafilter on a set I and λ be a cardinal. \mathfrak{u} is said to be λ -*descendingly complete* iff $\bigcap \{X_\xi : \xi < \lambda\} \in \mathfrak{u}$ for each decreasing sequence $\{X_\xi : \xi < \lambda\} \subset \mathfrak{u}$. The ultrafilter \mathfrak{u} is called λ -*descendingly incomplete* iff it is not λ -descendingly complete. For example, \mathfrak{u} is countably complete exactly if it is ω -descendingly complete.

We shall need the following old result of Kunen and Prikry in our next irresolvability theorem for spaces obtained from certain ultrafiltrations.

Theorem (Kunen, Prikry, [26]). *If λ is a regular cardinal and \mathfrak{u} is a λ -descendingly complete ultrafilter (on any set) then \mathfrak{u} is also λ^+ -descendingly complete.*

Theorem 4.14. *Assume that F is an ultrafiltration on T and λ is a regular cardinal such that $F(t)$ is λ -descendingly complete for all $t \in T$. Then the space $X(F)$*

is hereditarily λ^+ -irresolvable (that is, no subspace of $X(F)$ is λ^+ -resolvable).

Proof. First we show that for every set $A \subset T$ we have $\overline{A} = C^\lambda(A)$. By lemma 4.13 it suffices to show that $C(C^\lambda(A)) = C^\lambda(A)$.

Assume, indirectly, that $t \in C(C^\lambda(A)) \setminus C^\lambda(A)$, then we must have $C^\lambda(A) \cap S_t \in F(t)$. But

$$C^\lambda(A) \cap S_t = \bigcup_{\alpha < \lambda} C^\alpha(A) \cap S_t$$

where the right-hand side is an increasing union, hence there is an $\alpha < \lambda$ with $C^\alpha(A) \cap S_t \in F(t)$ because $F(t)$ is λ -descendingly complete. This implies that $t \in C^{\alpha+1}(A) \subset C^\lambda(A)$, a contradiction.

Let us now consider an *indexed* family of sets $\mathcal{F} = \{F_i : i \in I\}$. We are going to use the following notation:

$$\text{ord}(x, \mathcal{F}) = |\{i \in I : x \in F_i\}|$$

and

$$\text{ord}(\mathcal{F}) = \sup\{\text{ord}(x, \mathcal{F}) : x \in \bigcup_{i \in I} F_i\}.$$

Instead of the statement of the theorem we shall prove the following much stronger claim.

Lemma 4.15. *If $\mathcal{D} = \{D_i : i \in I\}$ is any indexed family of subsets of T with $\text{ord}(\mathcal{D}) \leq \lambda$ then $\text{ord}(\{\overline{D_i} : i \in I\}) \leq \lambda$ as well.*

Proof. We shall prove, by induction on $\alpha \leq \lambda$, that $\text{ord}(\mathcal{D}^\alpha) \leq \lambda$ where

$$\mathcal{D}^\alpha = \{C^\alpha(D_i) : i \in I\}.$$

We first show that $\text{ord}(\mathcal{D}^1) \leq \lambda$, this will clearly take care of all the successor steps.

Assume, indirectly, that $\text{ord}(t, \mathcal{D}^1) \geq \lambda^+$ for some $t \in T$, then we may find a set $J \in [I]^{\lambda^+}$ such that $t \in C(D_j) \setminus D_j$, hence $D_j \cap S_t \in F(t)$, for each $j \in J$.

By the theorem of Kunen and Prikry the ultrafilter $F(t)$ is also λ^+ -descendingly complete. Consequently, using a standard argument, one can show that there is an $L \in [J]^{\lambda^+}$ such that

$$\bigcap \{D_j \cap S_t : j \in L\} \neq \emptyset.$$

But this clearly contradicts $\text{ord}(\mathcal{D}) \leq \lambda$.

Next assume that $\alpha \leq \lambda$ is a limit ordinal and the inductive hypothesis holds for all $\beta < \alpha$. But now for each index $i \in I$ we have $C^\alpha(D_i) = \bigcup_{\beta < \alpha} C^\beta(D_i)$, hence

$$\text{ord}(t, \mathcal{D}^\alpha) \leq \sum_{\beta < \alpha} \text{ord}(t, \mathcal{D}^\beta) \leq |\alpha| \cdot \lambda = \lambda$$

whenever $t \in T$, and so $\text{ord}(\mathcal{D}^\alpha) \leq \lambda$. \square

It follows immediately from lemma 4.15 that if $\{A_i : i \in \lambda^+\}$ are pairwise disjoint non-empty subsets of T then the closures $\overline{A_i}$ cannot all be the same and so no subspace of $X(F)$ can be λ^+ -resolvable. \square

Corollary 4.16. *If F is an ultrafiltration on T such that $F(t)$ is countably complete for each $t \in T$ then $X(F)$ is ω -resolvable but hereditarily ω_1 -irresolvable. In particular, if κ is a measurable cardinal then there is a monotonically normal space X with $|X| = \Delta(X) = \kappa$ that is hereditarily ω_1 -irresolvable.*

The question if ω -resolvable spaces are also maximally resolvable was raised a long time ago by Ceder and Pearson in [9], and has just recently been settled completely in [23] (negatively). Corollary 4.16 yields a monotonically normal counterexample to this problem, from a measurable cardinal. Another counterexample from a measurable cardinal was found by Eckertson in [11], however, that example is not monotonically normal. We present two arguments to show this. First, Eckertson's example contains a crowded irresolvable subspace, hence it cannot be monotonically normal by corollary 4.4.

The second argument is based on our following observation that may have some independent interest. First we need some notation. If $\kappa \leq \lambda$ are cardinals we let τ_κ^λ denote the $< \kappa$ box product topology on 2^λ (generated by the base $\{[f] : f \in Fn(\lambda, 2; \kappa)\}$, where $[f] = \{x \in 2^\lambda : f \subset x\}$), moreover we set $\mathbb{C}_{\lambda, \kappa} = \langle 2^\lambda, \tau_\kappa^\lambda \rangle$.

Theorem 4.17. *If $\kappa^{<\kappa} = \kappa < \lambda$ then no dense subspace of $\mathbb{C}_{\lambda, \kappa}$ is monotonically normal.*

Proof of 4.17. Let X be dense in $\mathbb{C}_{\lambda, \kappa}$ and θ be a large enough regular cardinal. Let \mathcal{M} be an elementary submodel of $\langle \mathcal{H}(\theta), \in, \prec \rangle$ (where $\mathcal{H}(\theta)$ is the family of sets hereditarily of size $< \theta$ and \prec is a well-ordering of $\mathcal{H}(\theta)$) such that $|\mathcal{M}| = \kappa$ and $[\mathcal{M}]^{<\kappa} \subset \mathcal{M}$, moreover $X, \kappa, \lambda \in \mathcal{M}$. Note that then $Fn([\mathcal{M} \cap \lambda]^{<\kappa}, 2; \kappa) \subset \mathcal{M}$ as well.

Assume that X is monotonically normal and let $H \in \mathcal{M}$ be a monotone normality operator on X . We can assume that $H(x, [s] \cap X)$ is the trace on X of a

basic open set for each basic open set $[s]$.

Let $I = \mathcal{M} \cap \lambda$ and pick $\alpha \in \lambda \setminus I$. $\mathcal{F} = \{f \upharpoonright I : f \in \mathcal{M} \cap X\}$ is clearly dense in the subspace 2^I of $\mathbb{C}_{\lambda, \kappa}$. Let $\mathcal{F}_i = \{f \upharpoonright I : f \in X \cap \mathcal{M} \wedge f(\alpha) = i\}$ for $i \in 2$ then $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ so there is $i \in 2$ and $s \in Fn(I, 2; \kappa)$ such that \mathcal{F}_i is dense in $2^I \cap [s] \cap X$.

Let $b = s \cup \{\langle \alpha, 1 - i \rangle\}$ and pick $x \in X \cap [b]$. Next, let $H(x, [b] \cap X) = [b'] \cap X$ and $b'' = b' \upharpoonright I$. Fix $b''' \in Fn(I, 2; \kappa)$ such that $b''' \supset b''$ and $x \notin [b''']$. Since \mathcal{F}_i is dense in $2^I \cap [s] \cap X$ we can pick $y \in X \cap \mathcal{M} \cap [b''']$ such that $y(\alpha) = i$. Let $[u] \cap X = H(y, [b'''] \cap X)$. Then $\text{dom } u \subset I$ because $H, b''', y \in \mathcal{M}$.

Since $x \notin [b''']$ and $y \notin [b]$ it follows that $H(x, [b]) \cap H(y, [b''']) = [u] \cap [b'] \cap X = \emptyset$. However $\text{supp } u \subset I$ and $u \supset b''' \supset b'' = b' \upharpoonright I$, so u and b' are compatible functions of size $< \kappa$, i.e. $[u] \cap [b']$ is a nonempty open set in $\langle 2^\lambda, \tau_\kappa^\lambda \rangle$. Since X is dense we have $[u] \cap [b'] \cap X \neq \emptyset$, a contradiction. \square

Now, Eckertson's example obtained from a measurable cardinal κ contains a subspace homeomorphic to a dense subspace of $\mathbb{C}_{2^\kappa, \kappa}$, hence it cannot be monotonically normal by theorem 4.17 because $\kappa^{<\kappa} = \kappa$.

Of course, we have a space like in corollary 4.16 iff there is a measurable cardinal. Also, the cardinality (and dispersion character) of such a space is at least as large as the first measurable. But can we have an example of a monotonically normal and not maximally resolvable space that is much smaller? The answer to this question is, consistently, affirmative but, ironically, it requires the existence of a large cardinal that is much stronger than a measurable.

Theorem (Magidor, [30]). *It is consistent from a supercompact cardinal that there is an ω_1 -descendingly complete uniform ultrafilter on \aleph_ω .*

We would like to emphasize that in [4] a slightly weaker result was given in which \aleph_ω is replaced with $\aleph_{\omega+1}$. However, Magidor pointed out to us that the method of [4] yields the above stronger version as well. From Magidor's theorem and from theorem 4.14 we immediately obtain our promised result.

Corollary 4.18. *From a supercompact cardinal it is consistent to have a monotonically normal space X with $|X| = \Delta(X) = \aleph_\omega$ that is hereditarily ω_2 -irresolvable (hence not maximally resolvable).*

Of course, from [4] we could conclude the slightly weaker result in which \aleph_ω is replaced with $\aleph_{\omega+1}$.

But can we do even better and go below \aleph_ω ? The answer to this question is, maybe surprisingly, negative. We are going to show that any monotonically normal space of cardinality less than \aleph_ω is maximally resolvable. The proof of this result will be based on showing that all spaces of the form $X(F)$ with F an ultrafiltration on the tree $\text{Seq } \kappa = \kappa^{<\omega}$ of all finite sequences of ordinals less than κ are maximally resolvable provided that $\kappa < \aleph_\omega$. The first result to this effect, for constant ultrafiltrations on $\text{Seq } \omega_n$, was obtained by László Hegedüs in his Master's Thesis [18]. Of course, by a constant ultrafiltration we mean one for which $F(t)$ is the "same" ultrafilter for all $t \in T$.

Now, let κ be an arbitrary infinite cardinal. A non-empty subset T of $\text{Seq } \kappa$ is called a *subtree* of $\text{Seq } \kappa$ iff $t \upharpoonright n \in T$ whenever $t \in T$ and $n < |t|$. For any subset

A of $\text{Seq } \kappa$ we shall write $\min A$ to denote the set of all minimal elements of A (with respect to the tree ordering on $\text{Seq } \kappa$, of course).

If F is a filtration on $\text{Seq } \kappa$ and $v \in \text{Seq } \kappa$ we shall denote by F_v the derived filtration on $\text{Seq } \kappa$ defined by the formula $F_v(s) = F(v \hat{\ } s)$.

Assume now that S and $\{T_v : v \in \text{Seq } \kappa\}$ are subtrees of $\text{Seq } \kappa$. We then define their "sum" by

$$S \oplus \{T_v : v \in \text{Seq } \kappa\} = S \cup \{v \hat{\ } t : v \in \min(\text{Seq } \kappa \setminus S) \wedge t \in T_v\}.$$

Obviously, this sum is again a subtree of $\text{Seq } \kappa$.

If moreover f and $g = \{g_v : v \in \text{Seq } \kappa\}$ are functions with $\text{dom } f = S$ and $\text{dom } g_v = T_v$ then we define $f \oplus \{g_v : v \in \text{Seq } \kappa\} = f \oplus g$ by putting

$$\text{dom}(f \oplus g) = S \oplus \{T_v : v \in \text{Seq } \kappa\}$$

and

$$(f \oplus g)(x) = \begin{cases} f(x) & \text{for } x \in S \\ g_v(t) & \text{for } x = v \hat{\ } t \text{ with } v \in \min(\text{Seq } \kappa \setminus S), t \in T_v. \end{cases}$$

A subtree of $\text{Seq } \kappa$ is called *well-founded* iff it does not possess any infinite branches. Note that if S and $\{T_v : v \in \text{Seq } \kappa\}$ are all well-founded then so is $S \oplus \{T_v : v \in \text{Seq } \kappa\}$.

Now let $0 < \lambda \leq \kappa$ be cardinals and F be a filtration on $\text{Seq } \kappa$. We say that a function f is λ -good for F iff $\text{dom } f$ is a well-founded subtree of $\text{Seq } \kappa$, moreover $f[V] = \lambda$ whenever V is open in $X(F)$ with $\emptyset \in V$. As an easy (but useful) illustration of this concept we present the following result.

Lemma 4.19. *For each $0 < n < \omega$ and for any filtration F on κ there is a function f which is n -good for F .*

Proof. Let $\text{dom } f = \{s \in \text{Seq } \kappa : |s| < n\}$ and $f(s) = |s|$. □

The next result shows the relevance of these concepts to resolvability.

Theorem 4.20. *Let F be an filtration on $\text{Seq } \kappa$. If there are λ -good functions f_s for F_s for all $s \in \text{Seq } \kappa$ then $X(F)$ is λ -resolvable.*

Proof. Define the sequence of functions g_0, g_1, \dots by recursion as follows: $g_0 = f_\emptyset$ and $g_{n+1} = g_n \oplus \{f_s : s \in \text{Seq } \kappa\}$ for $n < \omega$. It is easy to check that then $g_\omega = \bigcup_{n < \omega} g_n$ maps $\text{Seq } \kappa$ to λ , i. e. $\text{dom } g_\omega = \text{Seq } \kappa$. Indeed, if $s \in \text{Seq } \kappa$ with $|s| = n$ then there is a $k \leq n$ with $s \in \text{dom } g_k$.

We show next that $g_\omega[V] = \lambda$ holds for any non-empty open set V in $X(F)$. Let n be such that $V \cap \text{dom } g_n \neq \emptyset$ and pick $v \in V \cap \text{dom } g_n$. Clearly, there is an extension s of v with $s \in V \cap \min(\text{Seq } \kappa \setminus \text{dom } g_n)$. Now let

$$W = \{t \in \text{Seq } \kappa : s \frown t \in V\}$$

then $\emptyset \in W$ and W is open in $X(F_s)$, hence $f_s[W] = \lambda$ because f_s is λ -good for F_s . But we clearly have $g_\omega(s \frown t) = f_s(t)$ for all $t \in \text{dom } f_s$, hence we have $g_\omega[V] = \lambda$ as well.

But then $\{g_\omega^{-1}(\alpha) : \alpha < \lambda\}$ is a pairwise disjoint family of dense sets in $X(F)$. □

The following stepping-up type result will turn out to be very useful.

Lemma 4.21. *Assume that F is a filtration on $\text{Seq } \kappa$ such that $F(\emptyset)$ is λ -descendingly incomplete, moreover for every cardinal $\mu < \lambda$ and every ordinal $\alpha < \kappa$ there is a μ -good function f_μ^α for $F_{\langle \alpha \rangle}$. Then there is a λ -good function f for F .*

Proof. Fix a continuously decreasing sequence $\{X_\xi : \xi < \lambda\} \subset F(\emptyset)$ with empty intersection. For any ordinal $\nu < \lambda$ let us put $I_\nu = X_\nu \setminus X_{\nu+1}$, then we clearly have $\kappa = \bigcup \{I_\nu : \nu < \lambda\}$. For each $0 < \nu < \lambda$ fix a map $h_\nu : |\nu| \xrightarrow{\text{onto}} \nu$.

We now define the desired map f with the following stipulations:

$$\text{dom } f = \{\emptyset\} \cup \bigcup_{\nu < \lambda} \{\langle \alpha \rangle \frown t : \alpha \in I_\nu \text{ and } t \in \text{dom } f_{|\nu|}^\alpha\},$$

and for $s \in \text{dom } f$

$$f(s) = \begin{cases} 0 & \text{if } s = \emptyset, \\ h_\nu(f_{|\nu|}^\alpha(t)) & \text{if } s = \langle \alpha \rangle \frown t \text{ with } \alpha \in I_\nu, t \in \text{dom } f_{|\nu|}^\alpha. \end{cases}$$

Clearly, f is well-defined and $\text{dom } f$ is well-founded. If V is open in $X(F)$ with $\emptyset \in V$ then we have $V \cap S_\emptyset \in F(\emptyset)$ and hence

$$\sup\{\nu : \exists \alpha \in I_\nu \text{ with } \langle \alpha \rangle \in V\} = \lambda.$$

But $\langle \alpha \rangle \in V$ and $\alpha \in I_\nu$ imply $f_{|\nu|}^\alpha[\{s : \langle \alpha \rangle \frown s \in V\}] = |\nu|$ and so $f[V] \supset \nu$, hence we have $f[V] = \lambda$. \square

Theorem 4.22. *Let F be a filtration on $\text{Seq } \kappa$ and λ be an infinite cardinal such that $F(t)$ is μ -descendingly incomplete whenever $t \in \text{Seq } \kappa$ and $\omega \leq \mu \leq \lambda$. Then there are λ -good functions for all the derived filtrations F_s and hence $X(F)$ is λ -resolvable.*

Proof. The proof goes by a straight-forward induction on λ , using lemma 4.21 and the fact that our assumption on F is automatically valid also for all the derived filtrations F_s . The starting case $\lambda = \omega$ also uses lemma 4.19. The last statement is immediate from theorem 4.20. \square

A uniform ultrafilter on κ is trivially κ -descendingly incomplete. So if $\kappa = \omega_n < \aleph_\omega$, then it follows by n repeated applications of the above mentioned result of Kunen and Prikry that any uniform ultrafilter on κ is μ -descendingly incomplete for all μ with $\omega \leq \mu \leq \kappa$. Thus we get from theorem 4.22 the following result.

Corollary 4.23. *Assume that $\kappa < \aleph_\omega$ and F is any uniform ultrafiltration on $\text{Seq } \kappa$ (i. e. $F(t)$ is uniform for all $t \in \text{Seq } \kappa$). Then the space $X(F)$ is κ -resolvable.*

We now recall a definition from [24], see also [31].

Definition 4.24. Let X be a space and μ be an infinite cardinal number. We say that $x \in X$ is a T_μ point of X if for every set $A \in [X]^{<\mu}$ there is some $B \in [X \setminus A]^{<\mu}$ such that $x \in \overline{B}$. We shall use $T_\mu(X)$ to denote the set of all T_μ points of X .

The following result is an easy consequence of lemma 1.3 from [24]. In the particular case when μ is a successor cardinal it follows from proposition 2.1 of [31].

Lemma 4.25. *If $|X| = \mu$ is a regular cardinal and $T_\mu(X)$ is dense in X then X is μ -resolvable.*

This result will enable us to transfer certain results from spaces of the form $X(F)$, where F is a uniform ultrafiltration on $\text{Seq } \kappa$ for some regular cardinal κ , to monotonically normal and even more general spaces.

Let us recall from section 1 that every monotonically normal space is SD. In fact, as monotone normality is a hereditary property, it is even hereditarily SD (in short: HSD). We shall need below a property that is strictly between SD and HSD, namely that all *dense* subspaces are SD, we shall denote this property by DSD. It can be shown that for instance the Čech-Stone remainder ω^* is DSD but not HSD.

Theorem 4.26. *Assume that $\kappa = \text{cf}(\kappa) \geq \lambda$. Then the following are equivalent.*

1. *If X is a DSD space with $|X| = \Delta(X) = \kappa$ then X is λ -resolvable.*
2. *If X is a MN space with $|X| = \Delta(X) = \kappa$ then X is λ -resolvable.*
3. *For every uniform ultrafiltration F on $\text{Seq } \kappa$ the space $X(F)$ is λ -resolvable.*

Proof. Of course, only (3) \Rightarrow (1) requires proof. So assume (3) and consider a DSD space X with $|X| = \Delta(X) = \kappa$. If $T_\kappa(X)$ is dense in X then, by lemma 4.25 X is even κ -resolvable and we are done.

Otherwise, in view of lemma 4.1, we may assume that actually $T_\kappa(X) = \emptyset$. In this case for every point $x \in X$ there is a set $A_x \in [X]^{<\kappa}$ such that $x \in A_x$ and for $D_x = X \setminus A_x$ no $B \in [D_x]^{<\kappa}$ has x in its closure. Note that by $\Delta(X) = \kappa$ each D_x is dense in X .

But X is DSD, hence for every x there is a strongly discrete set $S_x \subset D_x$ with $x \in \overline{S_x}$. (Note that $S \subset D_x$ is strongly discrete in D_x iff it is so in X for D_x is dense.)

Next, by recursion on $|t|$, we define points x_t and open sets U_t in X as follows. First pick any point $x_\emptyset \in X = U_\emptyset$. If $x_t \in U_t$ has been defined then fix a one-to-one enumeration of $S_{x_t} \cap U_t = \{x_{t \smallfrown \alpha} : \alpha < \kappa\}$ and choose $\{U_{t \smallfrown \alpha} : \alpha < \kappa\}$ to be pairwise disjoint open neighbourhoods of them, all contained in U_t . Clearly, then the map $h : \text{Seq } \kappa \longrightarrow X$ that maps t to $h(t) = x_t$ is injective.

Next, for any $t \in \text{Seq } \kappa$ extend the trace of the neighbourhood filter of x_t on $S_{x_t} \cap U_t$ to an ultrafilter u_t and define $F(t) = h^{-1}[u_t]$, which is an ultrafilter on $S_t = \{t \smallfrown \alpha : \alpha < \kappa\}$. It follows from our assumptions that every $F(t)$ is uniform and therefore $X(F)$ is λ -resolvable. But the subspace topology on $h[\text{Seq } \kappa]$ in X is clearly coarser than the h -image of τ_F , hence it is also λ -resolvable. By lemma 4.1, this completes our proof. \square

Corollary 4.27. *Let X be any DSD space of cardinality $< \aleph_\omega$. Then X is maximally resolvable. In particular, all MN spaces of size $< \aleph_\omega$ are maximally resolvable.*

Proof. Clearly, every open set U in X includes another open set V such that $|V| = \Delta(V)$. But every open subspace of a DSD space is again DSD, so theorem 4.26 and corollary 4.23 imply that V is $|V|$ -resolvable. But $\Delta(X) \leq |V|$, hence each such V is $\Delta(X)$ -resolvable and so, in view of lemma 4.1, X is maximally resolvable. \square

We conclude by listing a few open problems that we find especially interesting.

Problem 4.28.

1. *Is there a ZFC example of a monotonically normal space that is not maximally resolvable?*
2. *Is it consistent to have a monotonically normal space X of cardinality less than the first measurable such that $\Delta(X) > \omega$ but X is not ω_1 -resolvable?*
3. *Is every crowded monotonically normal space almost \mathfrak{c} -resolvable?*

Concerning problem (3) we have the following (very) partial result: Every countable crowded DSD space is almost \mathfrak{c} -resolvable.

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