Eigenvalue Problems for Some Elliptic Partial Differential Operators

by
Mihai Mihăilescu

Submitted to
Department of Mathematics and its Applications
Central European University

In partial fulfilment of the requirements for the degree of
Doctor of Philosophy in Mathematics and its Applications

Supervisor: Gheorghe Moroşanu

Budapest, Hungary
2010

Introduction

The concept of *eigenvalue* is basically related to *linear algebra* or *matrix theory* but it can be extended to differential operators. The study of eigenvalues has received much attention from many mathematicians throughout the years. We just remember that Euler, Lagrange, Cauchy, Hermite, Hilbert, Weierstrass, Fourier, Poincaré examined, among others, different eigenvalue problems.

In this thesis we will study eigenvalue problems associated with some elliptic partial differential operators. In a very general framework the *model* equations that will be considered here have one of the forms

$$-\mathrm{div}(\varphi(x,\nabla u)) = \lambda f(x,u) \tag{0.1}$$

or

$$-\sum_{i=1}^{N} \partial_{x_i}(\varphi_i(x,\partial_{x_i}u)) = \lambda f(x,u)\,, \tag{0.2}$$

where in the left-hand side we consider elliptic differential operators that can be linear or nonlinear, homogeneous or nonhomogeneous, while in the right-hand side λ is a real number and f is a given function. In this context, the concept of *eigenvalue* reads as follows: λ is an *eigenvalue* of problem (0.1) (or (0.2)) if the problem possesses a non-trivial solution u (here, solutions are understood in the sense of distributions).

Regarding the differential operators in the left-hand side of equations (0.1), (0.2), several important particular cases are included, such as: the *Laplace operator* (obtained if we take $\varphi(x,\nabla u) = \nabla u$ in (0.1)), the *p-Laplace operator* (obtained if we take $\varphi(x,\nabla u) = |\nabla u|^{p-2}\nabla u$ in (0.1) with $p \in (1,\infty)$, a given real number), the $p(x)$*-Laplace operator* (obtained if we take $\varphi(x,\nabla u) = |\nabla u|^{p(x)-2}\nabla u$ in (0.1) with $p(x) > 1$, a given continuous function) or *anisotropic operators* given by equation (0.2).

A short description of this thesis is presented in what follows. The thesis is divided into five chapters.

The first chapter is entitled *"Function spaces"*. The goal of this chapter is to offer a description of the variable exponent Lebesgue-Sobolev spaces and Orlicz-Sobolev spaces which are needed in the study of different eigenvalue problems that will be presented in the subsequent chapters.

The second chapter, entitled *"Eigenvalue problems involving the Laplace operator"*, comprises three sections. In the first section (based on paper [60]) an eigenvalue problem with a homogeneous Dirichlet boundary condition is analyzed. More exactly, in this section we highlight the case of an eigenvalue problem involving the Laplace operator which possesses, on the one hand, a continuous family of eigenvalues and, on the other hand, at least one more eigenvalue which is isolated in the set of eigenvalues of that problem. The second section (based on paper [50]) is devoted to the study of an eigenvalue problem on a bounded domain for the Laplace operator with a nonlinear Robin-like boundary condition. For that problem the existence, isolation and simplicity of the first two eigenvalues are proved. In the third section (based on paper [47]) we study an eigenvalue problem, involving a homogeneous Neumann boundary condition, in a smooth bounded domain. We show that the problem possesses, on the one hand, a continuous family of eigenvalues and, on the other hand, exactly one more eigenvalue which is isolated in the set of eigenvalues of the problem.

The third chapter is entitled *"Eigenvalue problems involving variable exponent growth conditions"* and comprises seven sections. The first part of section one remembers some known facts (obtained by X. Fan, Q. Zhang and D. Zhao in [30]) on the eigenvalue problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda |u|^{p(x)-2} u & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p : \overline{\Omega} \to (1, \infty)$ is a continuous function, $\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$ stands for the $p(x)$-Laplace operator and λ is a real number. The results in [30] are supplemented in the second part of this section by some new advances based on paper [63]. The second section of this chapter points out some known results on the eigenvalue problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda |u|^{q(x)-2} u & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, p, $q : \overline{\Omega} \to (1, \infty)$ are two continuous functions ($p \not\equiv q$). The third section (based on paper [65]) discusses an eigenvalue problem involving the $p(x)$-Laplace operator plus a non-local term. In this section the existence of a continuous family (an interval) of eigenvalues at the right of the origin is established. The forth section of this chapter is based on the results in [55]. More exactly, in this section the following boundary value problem is studied

$$\begin{cases} -\operatorname{div}((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}) \nabla u) = \lambda |u|^{q(x)-2} u & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary, λ is a positive real number, and the continuous functions p_1, p_2, and q satisfy $1 < p_2(x) < q(x) < p_1(x) < N$ and $\max_{y \in \overline{\Omega}} q(y) < \frac{N p_2(x)}{N - p_2(x)}$ for any $x \in \overline{\Omega}$. The main result of this section establishes the existence of two positive constants λ_0 and λ_1 with $\lambda_0 \leq \lambda_1$ such that any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue, while no $\lambda \in (0, \lambda_0)$ is an eigenvalue of the above problem. Next, in section five, an optimization result is presented in connection with a class of eigenvalue problems for which the problem in section four is a particular case. The results in section five are based on paper [61]. In section six an eigenvalue problem involving variable exponents is studied on an unbounded domain. The results in section six are based on paper [57]. The last section of chapter three is devoted to the study of some anisotropic eigenvalue problems involving variable growth conditions. The results therein are based on papers [53], [54], [49], [48].

The fourth chapter is entitled *"Eigenvalue problems in Orlicz-Sobolev spaces"* and is divided into four sections. In the first section the nonlinear eigenvalue problem

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = \lambda |u|^{q(x)-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is examined, where Ω is a bounded open set in $\mathbf{R}^N$ with smooth boundary, q is a continuous function, and a is a nonhomogeneous potential. Sufficient conditions on a and q are established such that the above nonhomogeneous quasilinear problem has continuous families of eigenvalues. The abstract results of this section are illustrated by the cases $a(t) = t^{p-2}\log(1+t^r)$ and $a(t) = t^{p-2}[\log(1+t)]^{-1}$. The results of this section are based on paper [59]. The second section of chapter four is devoted to the study of the boundary value problem

$$\begin{cases} -\operatorname{div}((a_1(|\nabla u| + a_2(|\nabla u|))\nabla u) = \lambda |u|^{q(x)-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$ ($N \geq 3$) with smooth boundary, λ is a positive real number, q is a continuous function and a_1, a_2 are two mappings such that $a_1(|t|)t$, $a_2(|t|)t$ are increasing homeomorphisms from $\mathbf{R}$ to $\mathbf{R}$. Sufficient conditions on a_1, a_2 and q are established such that for the above nonhomogeneous quasilinear problem there exist two positive real constants λ_0 and λ_1 with $\lambda_0 \leq \lambda_1$ such that every $\lambda \in [\lambda_1, \infty)$ is an eigenvalue of the above problem, while no $\lambda \in (0, \lambda_0)$ is an eigenvalue of the same problem. The results of this section are based on paper [58]. Next, in section three, an optimization result is presented in connection with a class of eigenvalue problems for which the problem in section two is a particular case. The results in section three are based on paper [62]. Finally, in section four we consider a class of anisotropic eigenvalue problems involving an elliptic, nonhomogeneous differential operator on a bounded domain of $\mathbf{R}^N$ with smooth boundary. Some results regarding the existence or non-existence of eigenvalues are obtained. In each case the competition between the growth rates of the anisotropic coefficients plays an essential role in the description of the set of eigenvalues. This section is based on the results in [51] and [52].

Finally, chapter five of this thesis is entitled *"Eigenvalue problems for difference equations"* and comprises two sections. The first section is based on paper [45]. In this section an eigenvalue problem is analyzed in the framework of difference equations. It is shown that there exist two positive constants λ_0 and λ_1 verifying $\lambda_0 \leq \lambda_1$ such that no $\lambda \in (0, \lambda_0)$ is an eigenvalue of the problem, while every $\lambda \in [\lambda_1, \infty)$ is an eigenvalue of the problem. Some estimates for λ_0 and λ_1 are also given. The second section of chapter five presents some results which are based on paper [64]. More exactly, in this section the existence of a continuous spectrum for a family of anisotropic discrete boundary value problems is established.

Acknowledgments

First, I want to express my deep gratitude to my supervisor, Professor Gheorghe Moroşanu, for accepting me as a Ph.D. student and for his guidance and professional advice during my Ph.D. studies at CEU. His personality and erudition have positively influenced my evolution and will accompany me for all my future career.

Furthermore, I am deeply indebted to my collaborators: Prof. Vicenţiu Rădulescu, Prof. Patrizia Pucci, Prof. Stepan Tersian, Prof. Dušan Repovš, Prof. Alexandru Kristály, Dr. Denisa Stancu-Dumitru. Without their cooperation my thesis would have undoubtedly been poorer. On a similar note, my appreciation equally goes to Prof. Alberto Farina for encouraging me to extend my research to anisotropic nonhomogeneous partial differential equations (PDE's). His bibliographical suggestions are highly appreciated.

Next, I wish to express my gratitude to the official referees of the dissertation. Their questions, suggestions and comments led to an improved version of this thesis.

I also warmly thank my colleagues from the Department of Mathematics and its Applications at Central European University for the academic atmosphere created within the department.

I owe my warmest gratitude to my good friend Liviu Ignat with whom I have spent many inspiring moments.

I am very grateful to my sister Dana for helping me in improving the language style of this manuscript.

A special thought goes to my mother for her never-ending support. This thesis is dedicated to her.

June 2010

Contents

1 Function spaces **1**
1.1 Variable exponent Lebesgue-Sobolev spaces 1
1.2 Orlicz-Sobolev spaces 4

2 Eigenvalue problems involving the Laplace operator **8**
2.1 An eigenvalue problem for the Laplace operator with the Dirichlet homogeneous boundary condition 8
2.2 An eigenvalue problem for the Laplace operator with a mixed nonlinear boundary condition 13
2.2.1 Introduction and main result 13
2.2.2 Proof of main result 15
2.2.3 Final comments 22
2.3 An eigenvalue problem for the Laplace operator with Neumann boundary condition . . . 24
2.3.1 Introduction and main results 24
2.3.2 Proof of Theorem 2.3 26
2.3.3 Proof of Theorem 2.4 30

3 Dirichlet eigenvalue problems involving variable exponent growth conditions **32**
3.1 Eigenvalue problem $-\Delta_{p(x)}u = \lambda|u|^{p(x)-2}u$ 32
3.2 Eigenvalue problem $-\Delta_{p(x)}u = \lambda|u|^{q(x)-2}u$ 35
3.3 An eigenvalue problem involving the $p(x)$-Laplace operator and a non-local term 36
3.4 Eigenvalue problem $-\Delta_{p_1(x)}u - \Delta_{p_2(x)}u = \lambda|u|^{q(x)-2}u$ 40
3.5 An optimization result 45
3.6 The case of unbounded domains 48
3.7 The anisotropic case 50

4 Dirichlet eigenvalue problems in Orlicz-Sobolev spaces **53**
4.1 Eigenvalue problem $-\text{div}(a(|\nabla u|)\nabla u) = \lambda|u|^{q(x)-2}u$ 53
4.1.1 Introduction 53
4.1.2 Proof of Theorem 4.1 55
4.1.3 Proof of Theorem 4.2 58
4.1.4 Examples 61

4.2 Eigenvalue problem $-\text{div}((a_1(|\nabla u|) + a_2(|\nabla u|))\nabla u) = \lambda |u|^{q(x)-2}u$ 62
4.2.1 Introduction 62
4.2.2 Auxiliary results 63
4.2.3 Main result 66
4.2.4 Proof of main result 67
4.3 An optimization result 72
4.4 The anisotropic case 75

5 Dirichlet eigenvalue problems for difference equations 78
5.1 Spectral estimates for a nonhomogeneous difference problem 78
5.1.1 Introduction and main results 78
5.1.2 Some estimates of eigenvalues 79
5.1.3 Proof of Theorem 5.1 84
5.2 Eigenvalue problems for anisotropic discrete boundary value problems 89

Chapter 1

Function spaces

In this chapter we introduce the definitions and basic properties of variable exponent Lebesgue-Sobolev spaces and Orlicz-Sobolev spaces. Even if both of these function spaces are particular cases of the so called Orlicz-Musielak spaces we prefer to introduce them separately in order to facilitate an easier understanding of their properties. For definitions and properties of Orlicz-Musielak spaces we refer to J. Musielak's [66] and M. Mihăilescu & V. Rădulescu's paper [56]. The classical Lebesgue and Sobolev spaces will be obtained as particular cases of the more general function spaces presented below.

1.1 Variable exponent Lebesgue-Sobolev spaces

In this section we provide a brief review of the basic properties of the variable exponent Lebesgue-Sobolev spaces. For more details we refer to the book by J. Musielak [66] and the papers by D. E. Edmunds *et al.* [21, 22, 23], O. Kovacik & J. Rákosník [43], and S. Samko & B. Vakulov [74].

In the following, let $\Omega \subset \mathbf{R}^N$ be an open set and denote by $|\Omega|$ the N-dimensional Lebesgue measure of the set Ω. For any Lipschitz continuous function $p : \overline{\Omega} \to (1, \infty)$ we denote

$$p^- = \text{ess inf}_{x\in\Omega} p(x) \quad \text{and} \quad p^+ = \text{ess sup}_{x\in\Omega} p(x)\,.$$

Usually it is assumed that $p^+ < +\infty$, since this condition is known to imply many desirable features for the associated variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$. This function space is defined by

$$L^{p(\cdot)}(\Omega) = \left\{u;\ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}.$$

On this space we define a norm, the so-called *Luxemburg norm*, by the formula

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)}\,dx \le 1\right\}.$$

The variable exponent Lebesgue space is a special case of an Orlicz-Musielak space. For constant functions p, $L^{p(\cdot)}(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$, endowed with the standard norm

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx\right)^{1/p}.$$

We recall that variable exponent Lebesgue spaces are separable and reflexive Banach spaces. If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents such that $p_1(x) \leq p_2(x)$ everywhere in Ω then there exists the continuous embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$.

We denote by $L^{p^{'}(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $1/p(x) + 1/p^{'}(x) = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p^{'}(\cdot)}(\Omega)$ the Hölder type inequality

$$\left| \int_\Omega uv \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(\cdot)} |v|_{p'(\cdot)} \tag{1.1}$$

holds true.

Moreover, if p_1, p_2, $p_3 : \overline{\Omega} \to (1, \infty)$ are three Lipschitz continuous functions such that $1/p_1(x) + 1/p_2(x) + 1/p_3(x) = 1$ then for any $u \in L^{p_1(\cdot)}(\Omega)$, $v \in L^{p_2(\cdot)}(\Omega)$ and $w \in L^{p_3(\cdot)}(\Omega)$ the following inequality holds (see [28, Proposition 2.5])

$$\left| \int_\Omega uvw \, dx \right| \leq \left(\frac{1}{p_1^-} + \frac{1}{p_2^-} + \frac{1}{p_3^-} \right) \, |u|_{p_1(\cdot)} \, |v|_{p_2(\cdot)} \, |w|_{p_3(\cdot)} \, . \tag{1.2}$$

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the *modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbf{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} \, dx.$$

Lebesgue–Sobolev spaces with $p^+ = +\infty$ have been investigated in [21, 43]. In such a case we denote $\Omega_\infty = \{x \in \Omega; \; p(x) = +\infty\}$ and define the modular by setting

$$\rho_{p(\cdot)}(u) = \int_{\Omega \setminus \Omega_\infty} |u(x)|^{p(x)} \, dx + \text{ess sup}_{x \in \Omega_\infty} |u(x)| \, .$$

If (u_n), $u \in L^{p(\cdot)}(\Omega)$ then the following relations hold true

$$|u|_{p(\cdot)} > 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p^+} \, , \tag{1.3}$$

$$|u|_{p(\cdot)} < 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p^-} \, , \tag{1.4}$$

$$|u_n - u|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u_n - u) \to 0 \, . \tag{1.5}$$

Next, we define the variable exponent Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \; |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

On $W^{1,p(\cdot)}(\Omega)$ we may consider one of the following equivalent norms

$$\|u\|_{p(\cdot)} = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}$$

or

$$\|u\| = \inf \left\{ \mu > 0; \; \int_\Omega \left(\left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} + \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) \, dx \leq 1 \right\} ,$$

where, in the definition of $\|u\|_{p(\cdot)}$, $|\nabla u|_{p(\cdot)}$ stands for the Luxemburg norm of $|\nabla u|$.

We also define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Assuming $p^- > 1$, then the function spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces. Set

$$\varrho_{p(\cdot)}(u) = \int_\Omega \left(|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)}\right) dx.$$

For all (u_n), $u \in W_0^{1,p(\cdot)}(\Omega)$ the following relations hold

$$\|u\| > 1 \;\Rightarrow\; \|u\|^{p^-} \le \varrho_{p(\cdot)}(u) \le \|u\|^{p^+}\,, \tag{1.6}$$

$$\|u\| < 1 \;\Rightarrow\; \|u\|^{p^+} \le \varrho_{p(\cdot)}(u) \le \|u\|^{p^-}\,, \tag{1.7}$$

$$\|u_n - u\| \to 0 \quad \Leftrightarrow \quad \varrho_{p(\cdot)}(u_n - u) \to 0\,. \tag{1.8}$$

We remember some embedding results regarding variable exponent Lebesgue–Sobolev spaces. If $p,\ q : \Omega \to (1,\infty)$ are Lipschitz continuous and $p^+ < N$ and $p(x) \le q(x) \le p^\star(x)$ for any $x \in \Omega$ where $p^\star(x) = Np(x)/(N - p(x))$, then there exists a continuous embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$. Furthermore, assuming that Ω_0 is a bounded subset of Ω, then the embedding $W_0^{1,p(\cdot)}(\Omega_0) \hookrightarrow L^{q(\cdot)}(\Omega_0)$ is continuous and compact, provided that $1 \le q(x) < p^\star(x)$ for any $x \in \Omega$, where $p^\star(x) = Np(x)/(N - p(x))$ if $p(x) < N$ and $p^\star(x) = \infty$ if $p(x) \ge N$. Furthermore, in this last case on the Sobolev space $W_0^{1,p(\cdot)}(\Omega_0)$ we can consider the equivalent norm

$$\|u\|_0 = |\nabla u|_{p(\cdot)}\,.$$

Finally, we consider the case when $\Omega \subset \mathbf{R}^N$ is open and bounded. In this case we introduce a natural generalization of the variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ that will enable us to study with sufficient accuracy problems involving anisotropic variable exponent operators. For this purpose, let us denote by $\overrightarrow{p} : \overline{\Omega} \to \mathbf{R}^N$ the vectorial function $\overrightarrow{p} = (p_1, ..., p_N)$, where $p_i : \overline{\Omega} \to (1,\infty)$ are continuous functions for each $i \in \{1, ..., N\}$. We define $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, the *anisotropic variable exponent Sobolev space*, as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{p}(\cdot)} = \sum_{i=1}^N |\partial_{x_i} u|_{p_i(\cdot)}\,.$$

As it was pointed out in [54], $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is a reflexive Banach space.

We also point out that in the case when $p_i : \overline{\Omega} \to (1,\infty)$ are constant functions for any $i \in \{1, .., N\}$ the resulting anisotropic Sobolev space is denoted by $W_0^{1,\overrightarrow{p}}(\Omega)$, where $\overrightarrow{p}$ is the constant vector $(p_1, ..., p_N)$. The theory of such spaces was developed in [32, 77, 69, 70, 76, 67].

On the other hand, in order to facilitate the manipulation of space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ we introduce $\overrightarrow{P}_+$, $\overrightarrow{P}_- \in \mathbf{R}^N$ as

$$\overrightarrow{P}_+ = (p_1^+, ..., p_N^+), \quad \overrightarrow{P}_- = (p_1^-, ..., p_N^-)\,,$$

and P_+^+, P_-^+, $P_-^- \in \mathbf{R}^+$ as

$$P_+^+ = \max\{p_1^+, ..., p_N^+\}, \quad P_-^+ = \max\{p_1^-, ..., p_N^-\}, \quad P_-^- = \min\{p_1^-, ..., p_N^-\}\,.$$

Here we always assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1, \tag{1.9}$$

and define $P_-^\star \in \mathbf{R}^+$ and $P_{-,\infty} \in \mathbf{R}^+$ by

$$P_-^\star = \frac{N}{\sum_{i=1}^N 1/p_i^- - 1}\,, \qquad P_{-,\infty} = \max\{P_-^+, P_-^\star\}\,.$$

We recall that if $s : \overline{\Omega} \to (1,\infty)$ is continuous and satisfies $1 < s(x) < P_{-,\infty}$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact (see [54, Theorem 1] or [53]).

1.2 Orlicz-Sobolev spaces

We first recall some basic facts about Orlicz spaces. For more details we refer to the books by D. R. Adams & L. L. Hedberg [4], R. Adams [3], J. Musielak [66] and M. M. Rao & Z. D. Ren [71] and the papers by Ph. Clément *et al.* [18, 19], M. Garciá-Huidobro *et al.* [33] and J. P. Gossez [37].

In the following, let $\Omega \subset \mathbf{R}^N$ be an open and bounded set and denote by $|\Omega|$ the N-dimensional Lebesgue measure of set Ω. Assume $\varphi : \mathbf{R} \to \mathbf{R}$ is an odd, increasing homeomorphism from $\mathbf{R}$ onto $\mathbf{R}$. Define

$$\Phi(t) = \int_0^t \varphi(s)\, ds, \quad \Phi^\star(t) = \int_0^t \varphi^{-1}(s)\, ds\,.$$

We observe that Φ is a *Young function*, that is, $\Phi(0) = 0$, Φ is convex, and $\lim_{t\to\infty} \Phi(t) = +\infty$. Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$, $\lim_{t\to 0} \Phi(t)/t = 0$, and $\lim_{t\to\infty} \Phi(t)/t = +\infty$, then Φ is called an *N–function.* Function $\Phi^\star$ is called the *complementary* function of Φ, and it satisfies

$$\Phi^\star(t) = \sup\{st - \Phi(s);\ s \geq 0\}, \qquad \text{for all } t \geq 0\,.$$

We also observe that $\Phi^\star$ is also an N–function and Young's inequality holds true

$$st \leq \Phi(s) + \Phi^\star(t), \qquad \text{for all } s, t \geq 0\,.$$

The Orlicz space $L_\Phi(\Omega)$ defined by the N–function Φ (see [4, 3, 18]) is the space of measurable functions $u : \Omega \to \mathbf{R}$ such that

$$\|u\|_{L_\Phi} := \sup\left\{\int_\Omega uv\, dx;\ \int_\Omega (\Phi)^\star(|g|)\, dx \leq 1\right\} < \infty\,.$$

Then $(L_\Phi(\Omega), \|\cdot\|_{L_\Phi})$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_\Phi := \inf\left\{k > 0;\ \int_\Omega \Phi\left(\frac{u(x)}{k}\right)\, dx \leq 1\right\}.$$

For Orlicz spaces Hölder's inequality reads as follows (see [71, Inequality 4, p. 79]):

$$\int_\Omega uvdx \leq 2\,\|u\|_{L_\Phi}\,\|v\|_{L_{\Phi^\star}} \qquad \text{for all } u\in L_\Phi(\Omega) \text{ and } v\in L_{\Phi^\star}(\Omega)\,.$$

We denote by $W^1L_\Phi(\Omega)$ the Orlicz-Sobolev space defined by

$$W^1L_\Phi(\Omega):=\left\{u\in L_\Phi(\Omega);\ \frac{\partial u}{\partial x_i}\in L_\Phi(\Omega),\ i=1,...,N\right\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{1,\Phi}:=\|u\|_\Phi+\||\nabla u|\|_\Phi\,.$$

We also define the Orlicz-Sobolev space $W_0^1L_\Phi(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^1L_\Phi(\Omega)$. By [37, Lemma 5.7] we obtain that on $W_0^1L_\Phi(\Omega)$ we may consider the equivalent norm

$$\|u\|_{0,\Phi}:=\||\nabla u|\|_\Phi\,.$$

Moreover, it can be proved that the above norm is equivalent with the following norm

$$\|u\|_{0,1,\Phi}=\sum_{j=1}^N\|\partial_j u\|_{\Phi_i}\,,$$

(see [52, Proposition 1] or [51]).

For an easier manipulation of Orlicz-Sobolev spaces we define

$$(p)_0:=\inf_{t>0}\frac{t\varphi(t)}{\Phi(t)} \ \text{ and } \ (p)^0:=\sup_{t>0}\frac{t\varphi(t)}{\Phi(t)}\,.$$

We assume that we have

$$1<(p)_0\leq\frac{t\varphi(t)}{\Phi(t)}\leq(p)^0<\infty,\quad \forall\, t\geq 0\,. \tag{1.10}$$

The above relation implies that Φ satisfies the Δ_2-condition, i.e.

$$\Phi(2t)\leq K\Phi(t),\quad \forall\, t\geq 0\,, \tag{1.11}$$

where K is a positive constant (see [56, Proposition 2.3]).

Furthermore, we assume that function Φ satisfies the following condition

$$\text{the function } [0,\infty)\ni t\to\Phi(\sqrt{t}) \text{ is convex}\,. \tag{1.12}$$

Conditions (1.11) and (1.12) assure that the Orlicz space $L_\Phi(\Omega)$ is a uniformly convex space and, thus, a reflexive Banach space (see [56, Proposition 2.2]). That fact implies that the Orlicz-Sobolev space $W_0^1L_\Phi(\Omega)$ is also a reflexive Banach space.

Examples. We point out certain examples of functions $\varphi:\mathbf{R}\to\mathbf{R}$ which are odd, increasing homeomorphisms from $\mathbf{R}$ onto $\mathbf{R}$ and satisfy conditions (1.10) and (1.12). For more details the reader can consult [19, Examples 1-3, p. 243].

1) Let

$$\varphi(t) = |t|^{p-2}t, \quad \forall\, t \in \mathbf{R}\,,$$

with $p > 1$. For this function it can be proved that

$$(p)_0 = (p)^0 = p\,.$$

2) Consider

$$\varphi(t) = \log(1 + |t|^r)|t|^{p-2}t, \quad \forall\, t \in \mathbf{R}\,,$$

with p, $r > 1$. In this case it can be proved that

$$(p)_0 = p, \qquad (p)^0 = p + r\,.$$

3) Let

$$\varphi(t) = \frac{|t|^{p-2}t}{\log(1 + |t|)}, \quad \text{if } t \neq 0, \quad \varphi(0) = 0\,,$$

with $p > 2$. In this case we have

$$(p)_0 = p - 1, \qquad (p)^0 = p\,.$$

Finally, we introduce a natural generalization of the Orlicz-Sobolev space $W_0^1 L_\Phi(\Omega)$ that will enable us to study with sufficient accuracy problems involving anisotropic differential operators. For this purpose, assume $\varphi_i : \mathbf{R} \to \mathbf{R}$, $i \in \{1, ..., N\}$, are odd, increasing homeomorphisms from $\mathbf{R}$ onto $\mathbf{R}$. Define

$$\Phi_i(t) = \int_0^t \varphi_i(s)\, ds, \qquad \text{for all } t \in \mathbf{R},\ i \in \{1, ..., N\}\,,$$

and

$$(p_i)_0 := \inf_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)} \ \text{ and } \ (p_i)^0 := \sup_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)},\ i \in \{1, ..., N\}\,.$$

Assume that φ_i and Φ_i satisfy conditions (1.10) and (1.12). Let us denote by $\overrightarrow{\Phi} : \overline{\Omega} \to \mathbf{R}^N$ the vectorial function $\overrightarrow{\Phi} = (\Phi_1, ..., \Phi_N)$, where $\Phi_i(t) = \int_0^t \varphi_i(s)\, ds$. We define $W_0^1 L_{\overrightarrow{\Phi}}(\Omega)$, the *anisotropic Orlicz-Sobolev space*, as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\overrightarrow{\Phi}} = \sum_{i=1}^N |\partial_i u|_{\Phi_i}\,.$$

It is natural to endow space $W_0^1 L_{\overrightarrow{\Phi}}(\Omega)$ with the norm $\|\cdot\|_{\overrightarrow{\Phi}}$ since Proposition 1 in [52] is valid. In the case when $\Phi_i(t) = |t|^{\theta_i}$, where θ_i are constants for any $i \in \{1, .., N\}$ the resulting anisotropic Sobolev space is denoted by $W_0^{1, \overrightarrow{\theta}}(\Omega)$, where $\overrightarrow{\theta}$ is the constant vector $(\theta_1, ..., \theta_N)$. The theory of such spaces was developed in [32, 77, 69, 70, 76, 67]. It was proved that $W_0^{1, \overrightarrow{\theta}}(\Omega)$ is a reflexive Banach space for any $\overrightarrow{\theta} \in \mathbf{R}^N$ with $\theta_i > 1$ for all $i \in \{1, .., N\}$. This result can be easily extended to $W_0^1 L_{\overrightarrow{\Phi}}(\Omega)$. Indeed, denoting by $X = L_{\Phi_1}(\Omega) \times ... \times L_{\Phi_N}(\Omega)$ and considering operator $T : W_0^1 L_{\vec{\Phi}}(\Omega) \to X$, defined by $T(u) = \nabla u$, it is clear that $W_0^1 L_{\vec{\Phi}}(\Omega)$ and X are isometric by T, since $\|Tu\|_X = \sum_{i=1}^N |\partial_i u|_{\Phi_i} = \|u\|_{\vec{\Phi}}$. Thus, $T(W_0^1 L_{\vec{\Phi}}(\Omega))$ is a closed subspace of X, which is a reflexive Banach space. By [13, Proposition III.17] it follows that $T(W_0^1 L_{\vec{\Phi}}(\Omega))$ is reflexive and, consequently, $W_0^1 L_{\vec{\Phi}}(\Omega)$ is a reflexive Banach space too.

On the other hand, in order to facilitate the manipulation of space $W_0^1 L_{\vec{\Phi}}(\Omega)$ we introduce $\vec{P^0}$, $\vec{P_0} \in \mathbf{R}^N$ as

$$\vec{P^0} = ((p_1)^0, ..., (p_N)^0), \quad \vec{P_0} = ((p_1)_0, ..., (p_N)_0),$$

and $(P^0)_+$, $(P_0)_+$, $(P_0)_- \in \mathbf{R}^+$ as

$$(P^0)_+ = \max\{(p_1)^0, ..., (p_N)^0\}, \quad (P_0)_+ = \max\{(p_1)_0, ..., (p_N)_0\}, \quad (P_0)_- = \min\{(p_1)_0, ..., (p_N)_0\}.$$

We assume that

$$\sum_{i=1}^N \frac{1}{(p_i)_0} > 1, \tag{1.13}$$

and define $P_0^\star \in \mathbf{R}^+$ and $P_{0,\infty} \in \mathbf{R}^+$ by

$$(P_0)^\star = \frac{N}{\sum_{i=1}^N 1/(p_i)_0 - 1}, \qquad P_{0,\infty} = \max\{(P_0)_+, (P_0)^\star\}.$$

Then, for any $q \in C(\overline{\Omega})$ verifying

$$1 < q(x) < P_{0,\infty} \quad \text{for all } x \in \overline{\Omega},$$

the embedding

$$W_0^1 L_{\vec{\Phi}}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

is compact (see, [52, Lemma 1] or [51]).

Chapter 2

Eigenvalue problems involving the Laplace operator

In this chapter we study three eigenvalue problems on bounded domains associated with the Laplace operator. We will assume that the first eigenvalue problem involves the Dirichlet homogeneous boundary condition, the second problem involves a nonlinear Robin boundary condition while the third problem involves the Neumann homogeneous boundary condition.

2.1 An eigenvalue problem for the Laplace operator with the Dirichlet homogeneous boundary condition

Throughout this section we assume that $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary. By an eigenvalue problem involving the Laplace operator with homogeneous boundary condition we understand a problem of the type

$$\begin{cases} -\Delta u = \lambda f(x,u), & \text{in} \quad \Omega \\ u = 0, & \text{on} \quad \partial\Omega\,, \end{cases} \tag{2.1}$$

where $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a given function and $\lambda \in \mathbf{R}$ is a real number. We say that λ is an *eigenvalue* of problem (2.1) if there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that for any $v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \nabla v \; dx - \lambda \int_\Omega f(x,u) v \; dx = 0\,.$$

Moreover, if λ is an eigenvalue of problem (2.1) then $u \in H_0^1(\Omega) \setminus \{0\}$ given in the above definition is called the *eigenfunction* corresponding to the eigenvalue λ. We are interested in finding positive eigenvalues for problems of type (2.1).

The study of eigenvalue problems involving the Laplace operator guides our mind back to a basic result in the elementary theory of partial differential equations which asserts that the problem below (which represents a particular case of problem (2.1), obtained when $f(x,u) = u$)

$$\begin{cases} -\Delta u = \lambda u, & \text{in} \quad \Omega \\ u = 0, & \text{on} \quad \partial\Omega\,, \end{cases} \tag{2.2}$$

possesses an unbounded sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq ... \leq \lambda_n \leq ....$ This celebrated result goes back to the Riesz-Fredholm theory of self-adjoint and compact operators on Hilbert spaces.

In what concerns λ_1, the lowest eigenvalue of problem (2.2), we remember that it can be characterized from a variational point of view as the minimum of the Rayleigh quotient, that is,

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx}{\displaystyle\int_\Omega u^2\, dx}\,. \tag{2.3}$$

Moreover, it is known that λ_1 is simple, that is, all the associated eigenfunctions are merely multiples of each other (see, e.g. Gilbarg and Trudinger [35]). Furthermore, the corresponding eigenfunctions of λ_1 never change signs in Ω.

Going further, another type of eigenvalue problems involving the Laplace operator (obtained in the case when we take in (2.1), $f(x,u) = |u|^{p-2}u$) is given by the nonlinear model equation

$$\begin{cases} -\Delta u = \lambda |u|^{p-2} u, & \text{in} \quad \Omega \\ u = 0, & \text{on} \quad \partial\Omega\,, \end{cases} \tag{2.4}$$

where $p \in (1, 2^*) \setminus \{2\}$ is a given real number and 2^* denotes the critical Sobolev exponent, that is,

$$2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geq 3 \\ +\infty & \text{if } N \in \{1,2\}. \end{cases}$$

Using a mountain-pass argument if $p > 2$ or the fact that the energy functional associated to problem (2.4) has a nontrivial (global) minimum point for any positive λ if $p < 2$, it can be proved that each $\lambda > 0$ is an eigenvalue of problem (2.4). Thus, in the case of problem (2.4) the set of eigenvalues consists of a continuous family, namely the interval $(0, \infty)$.

Motivated by the above results on problems (2.2) and (2.4) which show that the eigenvalue problems involving the Laplace operator can lead to a discrete spectrum (see the case of problem (2.2)) or a continuous spectrum (see the case of problem (2.4)) we consider it important to supplement the above situations by studying a new eigenvalue problem involving the Laplace operator which possesses, on the one hand, a continuous family of eigenvalues and, on the other hand, at least one more eigenvalue which is isolated in the set of eigenvalues of that problem.

We study problem (2.1) in the case when

$$f(x,t) = \begin{cases} h(x,t), & \text{if} \quad t \geq 0 \\ t, & \text{if} \quad t < 0\,, \end{cases} \tag{2.5}$$

where $h : \Omega \times [0, \infty) \to \mathbf{R}$ is a Carathéodory function satisfying the following hypotheses

(H1) there exists a positive constant $C \in (0, 1)$ such that $|h(x, t)| \leq Ct$ for any $t \geq 0$ and a.e. $x \in \Omega$;

(H2) there exists $t_0 > 0$ such that $H(x, t_0) := \int_0^{t_0} h(x, s)\, ds > 0$, for a.e. $x \in \Omega$;

(H3) $\lim_{t\to\infty} \frac{h(x,t)}{t} = 0$, uniformly in x.

Examples. We point out certain examples of functions h which satisfy hypotheses (H1)-(H3):

1. $h(x, t) = \sin(t/2)$, for any $t \geq 0$ and any $x \in \Omega$;
2. $h(x, t) = k \log(1 + t)$, for any $t \geq 0$ and any $x \in \Omega$, where $k \in (0, 1)$ is a constant;
3. $h(x, t) = g(x)(t^{q(x)-1} - t^{p(x)-1})$, for any $t \geq 0$ and any $x \in \Omega$, where p, $q : \overline{\Omega} \to (1, 2)$ are continuous functions satisfying $\max_{\overline{\Omega}} p < \min_{\overline{\Omega}} q$, and $g \in L^\infty(\Omega)$ satisfies $0 < \inf_\Omega g \leq \sup_\Omega g < 1$.

The main result of this section establishes a striking property of eigenvalue problem (2.1), provided that f is defined as in (2.5) and satisfies the above assumptions. More precisely, we prove that the first eigenvalue of the Laplace operator in $H_0^1(\Omega)$ is an *isolated* eigenvalue of (2.1) and, moreover, *any* λ sufficiently large is an eigenvalue, while the interval $(0, \lambda_1)$ does *not* contain any eigenvalue. This shows that problem (2.1) has both isolated eigenvalues and a continuous spectrum in a neighbourhood of $+\infty$.

Theorem 2.1. *Assume that f is given by relation (2.5) and conditions (H1), (H2) and (H3) are fulfilled. Then λ_1 defined in (2.3) is an isolated eigenvalue of problem (2.1) and the corresponding set of eigenvectors is a cone. Moreover, any $\lambda \in (0, \lambda_1)$ is not an eigenvalue of problem (2.1) but there exists $\mu_1 > \lambda_1$ such that any $\lambda \in (\mu_1, \infty)$ is an eigenvalue of problem (2.1).*

We notice that similar results as those given by Theorem 2.1 can be formulated for equations of type (2.6) if we replace the Laplace operator Δu by the p-Laplace operator, that is $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, with $1 < p < \infty$. Certainly, in that case hypotheses (H1)-(H3) should be modified according to the new situation. This statement is supported by the fact that the first eigenvalue of the p-Laplace operator on bounded domains satisfies similar properties as the one obtained in the case of the Laplace operator (see, e.g., [8]) combined with the remark that the results on problem (2.10) can be easily extended to the case of the p-Laplace operator.

Proof of Theorem 2.1. For any $u \in H_0^1(\Omega)$ we denote

$$u_\pm(x) = \max\{\pm u(x), 0\}, \quad \forall\, x \in \Omega\,.$$

Then u_+, $u_- \in H_0^1(\Omega)$ and

$$\nabla u_+ = \begin{cases} 0, & \text{if } [u \leq 0] \\ \nabla u, & \text{if } [u > 0]\,, \end{cases} \qquad \nabla u_- = \begin{cases} 0, & \text{if } [u \geq 0] \\ \nabla u, & \text{if } [u < 0]\,, \end{cases}$$

(see, e.g. [35, Theorem 7.6]). Thus, problem (2.1) with f given by relation (2.5) becomes

$$\begin{cases} -\Delta u = \lambda[h(x, u_+) - u_-], & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega\,, \end{cases} \tag{2.6}$$

and $\lambda > 0$ is an eigenvalue of problem (2.6) if there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla u_+ \nabla v \, dx - \int_\Omega \nabla u_- \nabla v \, dx - \lambda \int_\Omega [h(x,u_+) - u_-] v \, dx = 0\,, \tag{2.7}$$

for any $v \in H_0^1(\Omega)$.

Lemma 2.1. *Any $\lambda \in (0, \lambda_1)$ is not an eigenvalue of problem (2.6).*

Proof. Assume that $\lambda > 0$ is an eigenvalue of problem (2.6) with the corresponding eigenfunction u. Letting $v = u_+$ and $v = u_-$ in the definition of eigenvalue λ we find that the following two relations hold true

$$\int_\Omega |\nabla u_+|^2 \, dx = \lambda \int_\Omega h(x,u_+) u_+ \, dx \tag{2.8}$$

and

$$\int_\Omega |\nabla u_-|^2 \, dx = \lambda \int_\Omega u_-^2 \, dx\,. \tag{2.9}$$

In this context, hypothesis (H1) and relations (2.3), (2.8) and (2.9) imply

$$\lambda_1 \int_\Omega u_+^2 \, dx \le \int_\Omega |\nabla u_+|^2 \, dx = \lambda \int_\Omega h(x,u_+) u_+ \, dx \le \lambda \int_\Omega u_+^2 \, dx$$

and

$$\lambda_1 \int_\Omega u_-^2 \, dx \le \int_\Omega |\nabla u_-|^2 \, dx = \lambda \int_\Omega u_-^2 \, dx\,.$$

If λ is an eigenvalue of problem (2.6) then $u \neq 0$ and, thus, at least one of the functions u_+ and u_- is not the zero function. Thus, the last two inequalities show that λ is an eigenvalue of problem (2.6) only if $\lambda \ge \lambda_1$.

Lemma 2.2. *λ_1 is an eigenvalue of problem (2.6). Moreover, the set of eigenvectors corresponding to λ_1 is a cone.*

Proof. Indeed, as we have already pointed out, λ_1 is the lowest eigenvalue of problem (2.2), it is simple, that is, all the associated eigenfunctions are merely multiples of each other (see, e.g., Gilbarg and Trudinger [35]) and the corresponding eigenfunctions of λ_1 never change signs in Ω. In other words, there exists $e_1 \in H_0^1(\Omega) \setminus \{0\}$, with $e_1(x) < 0$ for any $x \in \Omega$ such that

$$\int_\Omega \nabla e_1 \nabla v \, dx - \lambda_1 \int_\Omega e_1 v \, dx = 0\,,$$

for any $v \in H_0^1(\Omega)$. Thus, we have $(e_1)_+ = 0$ and $(e_1) = -e_1$ and we deduce that relation (2.7) holds true with $u = e_1 \in H_0^1(\Omega) \setminus \{0\}$ and $\lambda = \lambda_1$. In other words, λ_1 is an eigenvalue of problem (2.6) and, undoubtedly, the set of its corresponding eigenvectors lies in a cone of $H_0^1(\Omega)$. The proof of Lemma 2.2 is complete.

Lemma 2.3. *λ_1 is isolated in the set of eigenvalues of problem (2.6).*

Proof. By Lemma 2.1 we know that in the interval $(0, \lambda_1)$ there is no eigenvalue of problem (2.6). On the other hand, hypothesis (H1) and relations (2.3) and (2.8) show that if λ is an eigenvalue of problem (2.6) for which the positive part of its corresponding eigenfunction, that is u_+, is not identically zero then

$$\lambda_1 \int_\Omega u_+^2 \, dx \leq \int_\Omega |\nabla u_+|^2 \, dx = \lambda \int_\Omega h(x, u_+) u_+ \, dx \leq \lambda C \int_\Omega u_+^2 \, dx \, ,$$

and, thus, since $C \in (0,1)$ we infer $\lambda \geq \frac{\lambda_1}{C} > \lambda_1$. We deduce that for any eigenvalue $\lambda \in (0, \lambda_1/C)$ of problem (2.6) we must have $u_+ = 0$. It follows that if $\lambda \in (0, \lambda_1/C)$ is an eigenvalue of problem (2.6) then it is actually an eigenvalue of problem (2.2) with the corresponding eigenfunction negative in Ω. Yet, we have already noticed that the set of eigenvalues of problem (2.2) is discrete and $\lambda_1 < \lambda_2$. In other words, taking $\delta = \min\{\lambda_1/C, \lambda_2\}$ we find that $\delta > \lambda_1$ and any $\lambda \in (\lambda_1, \delta)$ can not be an eigenvalue of problem (2.2) and, consequently, any $\lambda \in (\lambda_1, \delta)$ is not an eigenvalue of problem (2.6). We conclude that λ_1 is isolated in the set of eigenvalues of problem (2.6). The proof of Lemma 2.3 is complete.

Next, we show that there exists $\mu_1 > 0$ such that any $\lambda \in (\mu_1, \infty)$ is an eigenvalue of problem (2.6). With that end in view, we consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda h(x, u_+), & \text{in} \quad \Omega \\ u = 0, & \text{on} \quad \partial\Omega \, , \end{cases} \tag{2.10}$$

We say that λ is an eigenvalue of problem (2.10) if there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla u \nabla v \, dx - \lambda \int_\Omega h(x, u_+) v \, dx = 0 \, ,$$

for any $v \in H_0^1(\Omega)$.

We notice that if λ is an eigenvalue for (2.10) with the corresponding eigenfunction u, then taking $v = u_-$ in the above relation we deduce that $u_- = 0$, and thus, we find $u \geq 0$. In other words, the eigenvalues of problem (2.10) possess nonnegative corresponding eigenfunctions. Moreover, the above discussion shows that an eigenvalue of problem (2.10) is an eigenvalue of problem (2.6).

For each $\lambda > 0$ we define the energy functional associated to problem (2.10) by $I_\lambda : H_0^1(\Omega) \to \mathbf{R}$,

$$I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega H(x, u_+) \, dx \, ,$$

where $H(x,t) = \int_0^t h(x,s) \, ds$. Standard arguments show that $I_\lambda \in C^1(H_0^1(\Omega), \mathbf{R})$ with the derivative given by

$$\langle I_\lambda^{'}(u), v \rangle = \int_\Omega \nabla u \nabla v \, dx - \lambda \int_\Omega h(x, u_+) v \, dx \, ,$$

for any u, $v \in H_0^1(\Omega)$. Thus, $\lambda > 0$ is an eigenvalue of problem (2.10) if and only if there exists a critical nontrivial point of functional I_λ.

Lemma 2.4. *Functional I_λ is bounded from below and coercive.*

Proof. By hypothesis (H3) we deduce that

$$\lim_{t\to\infty}\frac{H(x,t)}{t^2}=0, \quad \text{uniformly in } \Omega\,.$$

Then for a given $\lambda>0$ there exists a positive constant $C_\lambda>0$ such that

$$\lambda H(x,t)\le \frac{\lambda_1}{4}t^2+C_\lambda, \quad \forall\, t\ge 0,\ \text{a.e. } x\in\Omega\,,$$

where λ_1 is given by relation (2.3).

Thus, we find that for any $u\in H^1_0(\Omega)$ the following inequality holds true

$$I_\lambda(u)\ge\frac{1}{2}\int_\Omega|\nabla u|^2\,dx-\frac{\lambda_1}{4}\int_\Omega u^2\,dx-C_\lambda|\Omega|\ge\frac{1}{4}\|u\|^2-C_\lambda|\Omega|\,,$$

where by $\|\cdot\|$ is denoted the norm on $H^1_0(\Omega)$, that is $\|u\|=(\int_\Omega|\nabla u|^2\,dx)^{1/2}$. This shows that I_λ is bounded from below and coercive. The proof of Lemma 2.4 is complete.

Lemma 2.5. *There exists $\lambda^\star>0$ such that assuming that $\lambda\ge\lambda^\star$ we have* $\inf_{H^1_0(\Omega)} I_\lambda<0$.

Proof. Hypothesis (H2) implies that there exists $t_0>0$ such that

$$H(x,t_0)>0 \quad \text{a.e. } x\in\overline{\Omega}\,.$$

Let $\Omega_1\subset\Omega$ be a compact subset, sufficiently large, and $u_0\in C^1_0(\Omega)\subset H^1_0(\Omega)$ such that $u_0(x)=t_0$ for any $x\in\Omega_1$ and $0\le u_0(x)\le t_0$ for any $x\in\Omega\setminus\Omega_1$.

Thus, by hypothesis (H1) we have

$$\begin{aligned}\int_\Omega H(x,u_0)\,dx &\ge \int_{\Omega_1}H(x,t_0)\,dx-\int_{\Omega\setminus\Omega_1}Cu_0^2\,dx\\ &\ge \int_{\Omega_1}H(x,t_0)\,dx-Ct_0^2|\Omega\setminus\Omega_1|>0\,.\end{aligned}$$

We conclude that $I_\lambda(u_0)<0$ for $\lambda>0$ sufficiently large, and thus, $\inf_{H^1_0(\Omega)} I_\lambda<0$. The proof of Lemma 2.5 is complete.

Lemmas 2.4 and 2.5 show that for any $\lambda>0$ large enough, functional I_λ possesses a negative global minimum (see, [75, Theorem 1.2]), and, thus, any $\lambda>0$ large enough is an eigenvalue of problem (2.10) and, consequently, of problem (2.6). Combining that fact with the results of Lemmas 2.1, 2.2 and 2.3 we conclude that Theorem 2.1 holds true.

2.2 An eigenvalue problem for the Laplace operator with a mixed nonlinear boundary condition

2.2.1 Introduction and main result

Assume $\Omega\subset\mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. We consider the following eigenvalue problem

$$\begin{cases}-\Delta u=\lambda u & \text{in}\quad \Omega,\\ -\dfrac{\partial u}{\partial\nu}=\alpha u_+ & \text{on } \partial\Omega\,.\end{cases}\tag{2.11}$$

where $\lambda \in \mathbf{R}$, $\partial u/\partial\nu$ denotes the outward normal derivative of u and $u_+(x) = \max\{u(x), 0\}$ for a.e. $x \in \Omega$.

The natural space for nonlinear eigenvalue problems of the type (2.11) is the Sobolev space $H^1(\Omega)$. Recall that if $u \in H^1(\Omega)$ then u_+, $u_- \in H^1(\Omega)$ and

$$\nabla u_+ = \begin{cases} 0, & \text{if } [u \le 0] \\ \nabla u, & \text{if } [u > 0], \end{cases} \qquad \nabla u_- = \begin{cases} 0, & \text{if } [u \ge 0] \\ \nabla u, & \text{if } [u < 0], \end{cases}$$

(see, e.g. [35, Theorem 7.6]), where $u_\pm(x) = \max\{\pm u(x), 0\}$ for a.e. $x \in \Omega$.

We will say that $\lambda \in \mathbf{R}$ is an *eigenvalue* of problem (2.11) if there exists $u \in H^1(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla u \nabla \varphi \, dx + \alpha \int_{\partial\Omega} u_+ \varphi \, d\sigma(x) = \lambda \int_\Omega u\varphi \, dx \,, \tag{2.12}$$

for any $\varphi \in H^1(\Omega)$. Such a function u will be called an *eigenfunction* corresponding to the eigenvalue λ. In fact, u is more regular. Indeed, it is known (see [10, Proposition 2.9, p. 63]) that $A = -\Delta$ with $D(A) = \{u \in H^2(\Omega);\ -\partial u/\partial\nu \in \beta(u)\ a.a.\ x \in \partial\Omega\}$ is a maximal (cyclically) monotone operator in $L^2(\Omega)$, and moreover there exist some constants C_1, $C_2 > 0$ such that

$$\|v\|_{H^2(\Omega)} \le C_1 \|v - \Delta v\|_{L^2(\Omega)} + C_2, \ \forall v \in D(A).$$

Therefore, if u is an eigenfunction of problem (2.11) corresponding to some λ, then it is easy to see that the (unique) solution of equation $v + Av = f$, where $f = (1 + \lambda)u$, is $v = u$, thus $u \in H^2(\Omega)$, and

$$\|u\|_{H^2(\Omega)} \le C_1 |1 + \lambda| \cdot \|u\|_{L^2(\Omega)} + C_2. \tag{2.13}$$

Note that u satisfies problem (2.11) in a classical sense.

Define

$$\lambda_1 = \inf_{v \in H^1(\Omega)\setminus\{0\},\ \int_\Omega v\, dx \ge 0} \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx + \alpha \int_{\partial\Omega} v_+^2 \, d\sigma(x)}{\displaystyle\int_\Omega v^2 \, dx} \,. \tag{2.14}$$

The main result of this section is given by the following theorem.

Theorem 2.2. *Numbers $\lambda_0 = 0$ and λ_1 (defined by relation (2.14)) represent the first two eigenvalues of problem (2.11), provided that $\alpha > 0$ is small. They are isolated in the set of eigenvalues of problem (2.11). Moreover, the sets of eigenfunctions corresponding to λ_0 and λ_1 are positive cones (more precisely, one-dimensional half-spaces) in $H^1(\Omega)$.*

The study of problem (2.11) is motivated by many applications. It is worth pointing out that we obtain in the next section a Rayleigh type principle: for $\alpha > 0$ small the first nontrivial eigenvalue λ_1 is a minimum value of the Rayleigh quotient associated with the corresponding classical Robin problem.

2.2.2 Proof of main result

Lemma 2.6. *No $\lambda < 0$ can be an eigenvalue of problem (2.11).*

Proof. Assume $\lambda \in \mathbf{R}$ is an eigenvalue of problem (2.11) with the corresponding eigenfunction $u \in H^1(\Omega) \setminus \{0\}$. Taking $\varphi = u$ in (2.12) we find

$$\lambda = \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx + \alpha \int_{\partial\Omega} u_+^2\,d\sigma(x)}{\displaystyle\int_\Omega u^2\,dx} \geq 0\,.$$

Lemma 2.7. *$\lambda_0 = 0$ is an eigenvalue of problem (2.11) and the set of its corresponding eigenfunctions is given by all the negative real constants.*

Proof. The first part of the lemma is obvious. Let us now consider $u \in H^1(\Omega) \setminus \{0\}$ an eigenfunction corresponding to λ_0. Taking $\varphi = u$ in relation (2.12) we deduce that

$$\int_\Omega |\nabla u|^2\,dx + \alpha \int_{\partial\Omega} u_+^2\,d\sigma(x) = 0\,.$$

Therefore, $\int_\Omega |\nabla u|^2\,dx = \int_{\partial\Omega} u_+^2\,d\sigma(x) = 0$. Consequently, u should be a negative real number.

Lemma 2.8. *λ_0 is isolated in the set of eigenvalues of problem (2.11).*

Proof. Assume by contradiction that λ_0 is not isolated. Then there exists a sequence of positive eigenvalues of problem (2.11), say (λ_n), such that $\lambda_n \searrow 0$. For each n we denote by u_n the corresponding eigenfunction of λ_n. Since we deal with a homogeneous problem we can assume that for each n we have $\|u_n\|_{L^2(\Omega)} = 1$. Relation (2.12) implies that for each n we have

$$\int_\Omega \nabla u_n \nabla\varphi\,dx + \alpha \int_{\partial\Omega} (u_n)_+\varphi\,d\sigma(x) = \lambda_n \int_\Omega u_n\varphi\,dx\,, \tag{2.15}$$

for any $\varphi \in H^1(\Omega)$. Taking $\varphi = u_n$ in relation (2.15) we find

$$\int_\Omega |\nabla u_n|^2\,dx + \alpha \int_{\partial\Omega} (u_n)_+^2\,d\sigma(x) = \lambda_n \int_\Omega u_n^2\,dx = \lambda_n\,. \tag{2.16}$$

We deduce that (u_n) is bounded in $H^1(\Omega)$. In fact, by estimate (2.13) with $\lambda = \lambda_n$ and $u := u_n$, it follows that (u_n) is bounded in $H^2(\Omega)$. Consequently, there exists $u \in H^2(\Omega)$ such that, on a subsequence, u_n converges strongly to u in $H^1(\Omega)$ and in $L^2(\partial\Omega)$ as well. Furthermore, $(u_n)_+$ converges strongly to u_+ in $L^2(\partial\Omega)$.

The above pieces of information lead to

$$\int_\Omega |\nabla u|^2\,dx + \alpha \int_{\partial\Omega} u_+^2\,d\sigma(x) = \lim_{n\to\infty} \left[\int_\Omega |\nabla u_n|^2\,dx + \alpha \int_{\partial\Omega} (u_n)_+^2\,d\sigma(x)\right] = \lim_{n\to\infty} \lambda_n = 0\,.$$

Thus, $\int_\Omega |\nabla u|^2\ dx = 0$ and $\int_{\partial\Omega} u_+^2\ d\sigma(x) = 0$. It follows that u is a negative constant satisfying $\|u\|_{L^2(\Omega)} = 1$. More precisely, $u = -1/|\Omega|^{1/2}$.

Turning back, relation (2.15) with $\varphi = u$ implies

$$\lambda_n \int_\Omega u_n u\ dx = -\alpha \frac{1}{|\Omega|^{1/2}} \int_{\partial\Omega} (u_n)_+\ d\sigma(x) \leq 0, \quad \text{for all } n\,.$$

It follows that

$$\int_\Omega u_n\ dx \geq 0, \quad \text{for all } n\,,$$

which implies

$$\int_\Omega u\ dx \geq 0\,.$$

This contradicts the fact that u is a negative constant. Consequently, the result of Lemma 2.8 holds true.

Remark 2.1. *Let us assume that $\lambda > 0$ is an eigenvalue of problem (2.11) with the corresponding eigenfunction u. Taking $\varphi \equiv 1$ in relation (2.12) it follows that*

$$\alpha \int_{\partial\Omega} u_+\ d\sigma(x) = \lambda \int_\Omega u\ dx\,,$$

which implies that

$$\int_\Omega u\ dx \geq 0\,.$$

Thus, the nonzero eigenvalues of problem (2.11) have the corresponding eigenfunctions in the cone

$$\mathcal{C} = \left\{ w \in H^1(\Omega);\ \int_\Omega w\ dx \geq 0 \right\}\,.$$

Consequently, the definition of λ_1 given in relation (2.14) is natural (we will prove later that for $\alpha > 0$ small enough λ_1 is an eigenvalue of problem (2.11)).

Lemma 2.9. *There exists $u \in \mathcal{C} \setminus \{0\}$ such that*

$$\lambda_1 = \frac{\displaystyle\int_\Omega |\nabla u|^2\ dx + \alpha \int_{\partial\Omega} u_+^2\ d\sigma(x)}{\displaystyle\int_\Omega u^2\ dx}\,.$$

Proof. Let $(u_n) \subset \mathcal{C} \setminus \{0\}$ be a minimizing sequence for λ_1, i.e.

$$\frac{\displaystyle\int_\Omega |\nabla u_n|^2\ dx + \alpha \int_{\partial\Omega} (u_n)_+^2\ d\sigma(x)}{\displaystyle\int_\Omega u_n^2\ dx} \to \lambda_1\,,$$

as $n \to \infty$. We can assume that $\|u_n\|_{L^2(\Omega)} = 1$ for all n. It follows that u_n is bounded in $H^1(\Omega)$. Thus, there exists $u \in H^1(\Omega)$ such that (a subsequence of) u_n converges weakly to u in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ and $L^2(\partial\Omega)$. It follows that $\|u\|_{L^2(\Omega)} = 1$, i.e. $u \neq 0$, and $\int_\Omega u\, dx \geq 0$. Thus, $u \in \mathcal{C} \setminus \{0\}$. The above pieces of information combined with the weak lower semicontinuity of the L^2-norm imply

$$\int_\Omega |\nabla u|^2\, dx + \alpha \int_{\partial\Omega} u_+^2\, d\sigma(x) \leq \lim_{n\to\infty} \left[\int_\Omega |\nabla u_n|^2\, dx + \alpha \int_{\partial\Omega} (u_n)_+^2\, d\sigma(x) \right] = \lambda_1 .$$

Since $\|u\|_{L^2(\Omega)} = 1$ the above inequality and the definition of λ_1 show that the conclusion of Lemma 2.9 holds true.

Remark 2.2. *We point out the fact that $\lambda_1 > 0$. Indeed, assuming by contradiction that $\lambda_1 = 0$ then by Lemma 2.9 there exists $u \in \mathcal{C} \setminus \{0\}$ such that*

$$\int_\Omega |\nabla u|^2\, dx + \int_{\partial\Omega} u_+^2\, d\sigma(x) = 0 .$$

It follows that u is a negative constant with $\int_\Omega u\, dx \geq 0$, a contradiction. Consequently $0 = \lambda_0 < \lambda_1$. Moreover, it is trivial to see that no $\lambda \in (0, \lambda_1)$ can be an eigenvalue of problem (2.11).

In the following we show that for $\alpha > 0$ small enough λ_1 is an eigenvalue of problem (2.11). In order to do that we denote for $\alpha \in (-\epsilon, \infty)$, with $\epsilon > 0$ small enough,

$$\lambda_1(\alpha) = \inf_{u \in \mathcal{C}\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx + \alpha \int_{\partial\Omega} u_+^2\, d\sigma(x)}{\displaystyle\int_\Omega u^2\, dx} ,$$

and

$$\mu_1(\alpha) = \inf_{u \in H^1(\Omega)\setminus\{0\},\ \int_\Omega u\, dx = 0} \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx + \alpha \int_{\partial\Omega} u_+^2\, d\sigma(x)}{\displaystyle\int_\Omega u^2\, dx} .$$

It is clear that for all $\alpha > 0$ we have $\mu_1(\alpha) \geq \lambda_1(\alpha)$, but, it is not obvious if either $\mu_1(\alpha) > \lambda_1(\alpha)$ or $\mu_1(\alpha) = \lambda_1(\alpha)$. However, we are able to prove the following result:

Lemma 2.10. *For any $\alpha > 0$ small enough we have $\mu_1(\alpha) > \lambda_1(\alpha)$.*

Proof. Obviously, for all $\alpha \geq 0$, both $\lambda_1(\alpha)$ and $\mu_1(\alpha)$ are finite. This property extends to $\alpha \in (-\epsilon, 0)$, with $\epsilon > 0$, small enough. Indeed, for all $u \in H^1(\Omega)$ with $\|u\|_{L^2(\Omega)} = 1$, we have (by the continuity of the trace operator)

$$\int_{\partial\Omega} u_+^2\, d\sigma(x) \leq \int_{\partial\Omega} u^2\, d\sigma(x) \leq C \left(\int_\Omega |\nabla u|^2\, dx + 1 \right) ,$$

where C is a positive constant. Therefore,

$$\int_\Omega |\nabla u|^2\, dx + \alpha \int_{\partial\Omega} u_+^2\, d\sigma(x) \geq (1 + \alpha C) \int_\Omega |\nabla u|^2\, dx + \alpha C \geq -\epsilon C ,$$

for all $\alpha \in (-\epsilon, 0)$, $u \in H^1(\Omega)$ with $\|u\|_{L^2(\Omega)} = 1$, provided that $\epsilon > 0$ satisfies $1 - \epsilon C \geq 0$. Thus, both $\lambda_1(\alpha)$ and $\mu_1(\alpha)$ are well defined for $\alpha \in (-\epsilon, \infty)$. (Moreover, a similar proof as the one used in Lemma 2.9 shows that both $\lambda_1(\alpha)$ and $\mu_1(\alpha)$ are attained.)

Now, let us point out the fact that functions $\lambda_1(\alpha)$, $\mu_1(\alpha) : (-\epsilon, \infty) \to \mathbf{R}$ are concave functions. Clearly, for any $\varphi \in \mathcal{C} \setminus \{0\}$ function

$$(-\epsilon, \infty) \ni \alpha \longrightarrow \frac{\displaystyle\int_\Omega |\nabla\varphi|^2\, dx + \alpha \int_{\partial\Omega} \varphi_+^2\, d\sigma(x)}{\displaystyle\int_\Omega \varphi^2\, dx},$$

is an affine function, consequently, a concave function. Since the infimum of a family of concave functions is a concave function, it follows that $\lambda_1(\alpha)$ is concave. Similarly, $\mu_1(\alpha)$ is also concave. Thus, we deduce that $\lambda_1(\alpha)$ and $\mu_1(\alpha)$ are continuous functions for $\alpha \in (-\epsilon, \infty)$. On the other hand, $\lambda_1(0) = 0$ and $\mu_1(0) = \lambda_{1,N}$, where 0 and $\lambda_{1,N}$ are the first two eigenvalues of the Neumann problem (see, e.g. [34, Chapter 4.2.1]), i.e.

$$\begin{cases} -\Delta u = \lambda u & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.17}$$

It is well-known that $\lambda_{1,N} > 0$ (see, [34, Proposition 4.2.2 and Proposition 4.2.3]). Thus, we found $\lambda_1(0) < \mu_1(0)$. This inequality and the fact that $\lambda_1(\alpha)$ and $\mu_1(\alpha)$ are continuous functions for $\alpha \in (-\epsilon, \infty)$ imply that $\lambda_1(\alpha) < \mu_1(\alpha)$ for any $\alpha > 0$, small enough. The proof of Lemma 2.10 is complete.

Lemma 2.11. *Assume that $u \in \mathcal{C} \setminus \{0\}$ is a minimizer for the infimum given by relation (2.14), with $\int_\Omega u\, dx > 0$. Then λ_1 is an eigenvalue of problem (2.11) and u is an eigenfunction corresponding to λ_1.*

Proof. Let $\varphi \in H^1(\Omega)$ be fixed. Then for any ϵ lying in a small neighborhood of the origin, we have $\int_\Omega (u + \epsilon\varphi)\, dx > 0$, i.e. $u + \epsilon\varphi \in \mathcal{C}$. Define function

$$f(\epsilon) = \frac{\displaystyle\int_\Omega |\nabla(u + \epsilon\varphi)|^2\, dx + \alpha \int_{\partial\Omega} (u + \epsilon\varphi)_+^2\, d\sigma(x)}{\displaystyle\int_\Omega (u + \epsilon\varphi)^2\, dx}.$$

Clearly, f is well defined in a small neighborhood of the origin and possesses a minimum in $\epsilon = 0$. Consequently,

$$f'(0) = 0\,,$$

or, by some simple computations,

$$\int_\Omega \nabla u \nabla\varphi\, dx + \alpha \int_{\partial\Omega} u_+\varphi\, d\sigma(x) = \lambda_1 \int_\Omega u\varphi\, dx\,.$$

Clearly the above equality holds true for any $\varphi \in H^1(\Omega)$. We deduce that u is an eigenfunction corresponding to eigenvalue λ_1, and the proof of Lemma 2.11 is complete.

Proposition 2.1. *Number λ_1, defined by relation (2.14), is an eigenvalue of problem (2.11), provided that $\alpha > 0$ is small enough.*

Proof. The conclusion of Proposition 2.1 is a simple consequence of Lemmas 2.9, 2.10 and 2.11.

Lemma 2.12. *If λ_1 is an eigenvalue of problem (2.11) and $u \in H^1(\Omega) \setminus \{0\}$ is an eigenfunction corresponding to λ_1, then $u \geq 0$ in Ω (thus, $\int_\Omega u\, dx > 0$).*

Proof. Relation (2.12) shows that

$$\int_\Omega \nabla u \nabla \varphi\, dx + \alpha \int_{\partial\Omega} u_+ \varphi\, d\sigma(x) = \lambda_1 \int_\Omega u\varphi\, dx\,, \tag{2.18}$$

for every $\varphi \in H^1(\Omega)$. First, we claim that $u_+ \neq 0$. Indeed, assuming the contrary, we deduce that

$$\int_\Omega \nabla u_- \nabla \varphi\, dx = \lambda_1 \int_\Omega u_- \varphi\, dx\,, \tag{2.19}$$

for every $\varphi \in H^1(\Omega)$. Taking $\varphi = 1$ we find

$$\int_\Omega u_-\, dx = 0\,,$$

that means, $u_- = 0$ and, thus, $u = 0$, a contradiction. Consequently, $u_+ \neq 0$. Then, taking $\varphi = u_+$ in (2.18) we have

$$\lambda_1 = \frac{\displaystyle\int_\Omega |\nabla u_+|^2\, dx + \alpha \int_{\partial\Omega} u_+^2\, d\sigma(x)}{\displaystyle\int_\Omega u_+^2\, dx}\,.$$

By Lemma 2.11 we infer that u_+ is an eigenfunction corresponding to λ_1, or

$$\int_\Omega \nabla u_+ \nabla \varphi\, dx + \alpha \int_{\partial\Omega} u_+ \varphi\, d\sigma(x) = \lambda_1 \int_\Omega u_+ \varphi\, dx\,, \tag{2.20}$$

for every $\varphi \in H^1(\Omega)$. Relations (2.18) and (2.20) imply that relation (2.19) holds true. Taking again $\varphi = 1$ in (2.19) we find again $\int_\Omega u_-\, dx = 0$ which leads to $u_- = 0$ in Ω. The proof of Lemma 2.12 is complete.

Remark 2.3. *By Lemma 2.12, if λ_1 is an eigenvalue of problem (2.11), then it is the first eigenvalue of the following Robin problem*

$$\begin{cases} -\Delta u = \lambda u & \text{in} \quad \Omega, \\ -\dfrac{\partial u}{\partial \nu} = \alpha u & \text{on } \partial\Omega\,. \end{cases} \tag{2.21}$$

In the following we argue that fact in detail. It is well-known that number

$$\gamma_1 = \inf_{v \in H^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla v|^2\, dx + \alpha \int_{\partial\Omega} v^2\, d\sigma(x)}{\displaystyle\int_\Omega v^2\, dx}\,,$$

known as the Rayleigh quotient, is positive and represents the first eigenvalue of problem (2.21). Moreover, γ_1 is simple, that means, all the associated eigenfunctions are merely multiples of each other. It is also known that these eigenfunctions belong to $C(\overline{\Omega}) \cap C^1(\Omega)$ (see [14, Lemma 2.1]). Furthermore, an eigenfunction of γ_1 can be chosen with a single sign, particularly with positive sign (see, e.g. [36]). The definitions of γ_1 and λ_1 show that $\gamma_1 \geq \lambda_1$. Actually, by Lemma 2.12 we have $\lambda_1 = \gamma_1$, i.e. λ_1 is the first eigenvalue of problem (2.21). Thus, the set of eigenfunctions corresponding to λ_1 is a positive cone in $H^1(\Omega)$. More precisely, if u is a positive eigenfunction for the Robin problem, associated with γ_1, then the set of eigenfunctions for problem (2.11), associated with λ_1 $(= \gamma_1)$, is the one dimensional half-space $\{tu; t > 0\}$. Hence λ_1 is simple.

Finally, we focus our attention on proving that λ_1 is isolated. We will use a technique borrowed from [8] that will be described in what follows.

Lemma 2.13. *Assume $\lambda > 0$ is an eigenvalue of problem (2.11) and $u \in H^1(\Omega)\backslash\{0\}$ is an eigenfunction corresponding to λ. Define $\Omega_- = \{x \in \Omega;\ \ u(x) < 0\}$. If $|\Omega_-| > 0$ then there exists a positive constant C (independent of λ and u) such that*

$$((\lambda + 1)C)^{-N/2} \leq |\Omega_-| \, .$$

Proof. Recalling again relation (2.12) we have

$$\int_\Omega \nabla u \nabla \varphi \, dx + \alpha \int_{\partial\Omega} u_+ \varphi \, d\sigma(x) = \lambda \int_\Omega u\varphi \, dx \, ,$$

for every $\varphi \in H^1(\Omega)$. Taking $\varphi = u_-$ we find

$$\int_\Omega |\nabla u_-|^2 \, dx = \lambda \int_\Omega u_-^2 \, dx \, ,$$

or by taking into account that $L^{2^\star}(\Omega)$ is continuously embedded in $L^2(\Omega)$, where $2^\star = 2N/(N-2)$ is the critical Sobolev exponent, we deduce by Hölder's inequality

$$\int_\Omega |\nabla u_-|^2 \, dx + \int_\Omega u_-^2 \, dx = (\lambda + 1) \int_\Omega u_-^2 \, dx \leq (\lambda + 1) \|u_-\|^2_{L^{p^\star}(\Omega)} |\Omega_-|^{1-2/2^\star} \, .$$

Next, since $H^1(\Omega)$ is continuously embedded in $L^{2^\star}(\Omega)$ we deduce that there exists a positive constant C such that

$$\|v\|^2_{L^{2^\star}(\Omega)} \leq C \left(\int_\Omega |\nabla v|^2 \, dx + \int_\Omega v^2 \, dx \right) \, ,$$

for any $v \in H^1(\Omega)$. The last two inequalities imply

$$1 \leq (\lambda + 1) C |\Omega_-|^{2/N} \, .$$

The proof of Lemma 2.13 is complete.

Lemma 2.14. *λ_1 is isolated in the set of eigenvalues of problem (2.11).*

Proof. By Remark 2.2 it is clear that λ_1 is isolated from the left. We show that it is also isolated from the right. Assume by contradiction that this is not the case. Then there exists a sequence of positive eigenvalues of problem (2.11), say (λ_n), such that $\lambda_n \searrow \lambda_1$. For each n we denote by u_n an eigenfunction corresponding to λ_n. Since we deal with a homogeneous problem we can assume that for each n we have $\|u_n\|_{L^2(\Omega)} = 1$. Relation (2.12) implies that for each n we have

$$\int_\Omega \nabla u_n \nabla \varphi \, dx + \alpha \int_{\partial\Omega} (u_n)_+ \varphi \, d\sigma(x) = \lambda_n \int_\Omega u_n \varphi \, dx \,, \tag{2.22}$$

for any $\varphi \in H^1(\Omega)$. Arguing as in the proof of Lemma 2.8, we deduce that (u_n) is bounded in $H^2(\Omega)$. Consequently, there exists $u \in H^2(\Omega)$ such that u_n converges, on a subsequence, to u in $H^1(\Omega)$ and in $L^2(\partial\Omega)$ as well. Furthermore, we also have $(u_n)_+$ converges strongly to u_+ in $L^2(\partial\Omega)$. Passing to the limit as $n \to \infty$ in (2.22) we get

$$\int_\Omega \nabla u \nabla \varphi \, dx + \alpha \int_{\partial\Omega} (u)_+ \varphi \, d\sigma(x) = \lambda_1 \int_\Omega u \varphi \, dx \,, \tag{2.23}$$

for any $\varphi \in H^1(\Omega)$. Since $\|u\|_{L^2(\Omega)}$=1 it follows that $u \neq 0$ and, thus, it is an eigenfunction corresponding to λ_1. By Lemma 2.12 we deduce that $u \geq 0$ in Ω. In fact, according to Remark 2.3, $u \in C(\overline{\Omega}) \cap C^1(\Omega)$ and $u(x) > 0$ for all $x \in \Omega$. Let now $\epsilon > 0$ be arbitrary but fixed and let $K \subset \Omega$ be a compact such that $|\Omega \setminus K| < \epsilon/2$. Obviously, there exists a $\delta > 0$ (depending on K) such that $u(x) \geq \delta > 0$ for every $x \in K$.

On the other hand, it is clear that u_n converges to u a.e. in Ω and thus, in K. Consequently, by Egorov's Theorem (see, e.g. [78, Théorème 2.37]) we deduce that for $\epsilon > 0$ fixed above there exists a measurable set $\omega \subset K$ with $|\omega| < \epsilon/2$ such that u_n converges uniformly to u on $K \setminus \omega$. Since $u \geq \delta > 0$ in K we deduce that for any n large enough we have $u_n \geq 0$ on $K \setminus \omega$. For each n we define $(\Omega_n)_- = \{x \in \Omega; \;\; u_n(x) < 0\}$. We can assume that for each n the fact that $|(\Omega_n)_-| > 0$ holds true. Indeed, otherwise, there exists a particular n for which we have $u_n \geq 0$ (and $u_n \neq 0$) in Ω. Taking $\varphi = u$ in (2.22) and $\varphi = u_n$ in (2.23) we deduce that

$$\lambda_n \int_\Omega u_n u \, dx = \lambda_1 \int_\Omega u u_n \, dx \,.$$

Since $\int_\Omega u u_n \; dx > 0$ the above equality leads to $\lambda_n = \lambda_1$ which represents a contradiction with the fact that $\lambda_n > \lambda_1$. Consequently, we should have $|(\Omega_n)_-| > 0$ for all n. It follows that for any n large enough we have $(\Omega_n)_- \subset \omega \cup (\Omega \setminus K)$. Using the above facts and Lemma 2.13 we have the following inequalities which hold true

$$((\lambda_n + 1)C)^{-N/2} \leq |(\Omega_n)_-| \leq |\omega| + |\Omega \setminus K| < \epsilon \,,$$

provided that n is large enough. Therefore,

$$((\lambda_1 + 1)C)^{-N/2} \leq \epsilon,$$

for all $\epsilon > 0$, which is impossible. Consequently, the conclusion of Lemma 2.14 holds true.

2.2.3 Final comments

In this section we point out some facts that are direct consequences of the discussion presented in the above sections.

First, we highlight the fact that for any $\alpha > 0$, number $\gamma_1 = \gamma_1(\alpha)$, introduced in Remark 2.3 and which represents the first eigenvalue of the Robin problem (that is problem (2.21)) is an eigenvalue of problem (2.11). The above assertion is a consequence of the fact that there exists $u \in H^1(\Omega) \setminus \{0\}$ with $u \geq 0$ a.e. in Ω such that

$$\int_\Omega \nabla u \nabla \varphi \, dx + \alpha \int_{\partial\Omega} u\varphi \, d\sigma(x) = \gamma_1 \int_\Omega u\varphi \, dx \,,$$

for all $\varphi \in H^1(\Omega)$. Since $u \geq 0$ a.e. in Ω it follows that, actually, relation (2.12) is verified in the case when $\lambda = \gamma_1$. The definitions of $\lambda_1(\alpha)$ and $\gamma_1(\alpha)$ imply that for any $\alpha > 0$ we have $\gamma_1(\alpha) \geq \lambda_1(\alpha)$. Moreover, by Remark 2.3 we know that for $\alpha > 0$ *small enough* we have $\gamma_1(\alpha) = \lambda_1(\alpha)$. However, we cannot conclude that for *any* $\alpha > 0$ we have $\gamma_1(\alpha) = \lambda_1(\alpha)$.

Second, we focus our attention on numbers $\lambda_1(\alpha)$ and $\mu_1(\alpha)$ defined in accordance with Remark 2.2. It is clear that for *all* $\alpha > 0$ we have $\mu_1(\alpha) \geq \lambda_1(\alpha)$. Moreover, for $\alpha > 0$ *small enough*, by Lemma 2.10, we have that $\mu_1(\alpha) > \lambda_1(\alpha)$ and $\lambda_1(\alpha)$ is an eigenvalue of problem (2.11) (see, Lemma 2.11). On the other hand, nothing is clear if $\alpha > 0$ is far from the origin. At least theoretically it may happen that for some $\alpha > 0$ large we have $\mu_1(\alpha) = \lambda_1(\alpha)$. In that case the reasoning from Lemma 2.11 does not work and consequently we cannot state whether $\lambda_1(\alpha)$ is an eigenvalue or not. However, we can show the following result which is undoubtedly connected with the above discussion:

Proposition 2.2. *If there exists $\alpha > 0$ for which any minimizer $u \in \mathcal{C} \setminus \{0\}$ of $\lambda_1(\alpha)$ satisfies $\int_\Omega u \, dx = 0$ then $\lambda_1(\alpha)$ is not an eigenvalue of problem (2.11).*

Proof. Assume, by contradiction, that $\lambda_1(\alpha)$ is an eigenvalue of problem (2.11). Then, any eigenfunction u corresponding to $\lambda_1(\alpha)$ is a minimizer with $\int_\Omega u \, dx = 0$. On the other hand, by Lemma 2.12 we have $\int_\Omega u \, dx > 0$, a contradiction. The proof of Proposition 2.2 is complete.

Define

$$V = \{u \in H^1(\Omega); \ \int_\Omega u \, dx = 0\} \,.$$

Clearly, $H^1(\Omega) = V \oplus \mathbf{R}$ and $V \subset \mathcal{C}$. It seems that for some $\alpha > 0$ large $\lambda_1(\alpha)$ is attained on V, i.e., $\lambda_1(\alpha) = \mu_1(\alpha)$. In this case, by Proposition 2, $\lambda_1(\alpha)$ is not an eigenvalue of problem (2.11). Since in general $\lambda_1 \leq \gamma_1$, we would have in this case $\lambda_1(\alpha) < \gamma_1(\alpha)$.

A similar proof as the one of Lemma 2.9 shows that for each $\alpha > 0$ there exists $v_\alpha \in V \setminus \{0\}$ a minimizer of $\mu_1(\alpha)$. Moreover, as in Lemma 2.11 it can be proved that for v_α given above we have

$$\int_\Omega \nabla v_\alpha \nabla \varphi \, dx + \alpha \int_{\partial\Omega} (v_\alpha)_+ \varphi \, d\sigma(x) = \mu_1(\alpha) \int_\Omega v_\alpha \varphi \, dx \,, \tag{2.24}$$

for all $\varphi \in V$. However, the above relation is not enough to state that $\mu_1(\alpha)$ is an eigenvalue of problem (2.11) in the sense of the definition given by relation (2.12).

In connection with the above discussion, let us introduce the following definition: we say that $\lambda > 0$ is an *extended eigenvalue* of problem (2.11) if there exists $u \in \mathcal{C} \setminus \{0\}$ such that

$$\int_\Omega \nabla u \nabla(\varphi - u)\, dx + \alpha \int_{\partial\Omega} u_+(\varphi - u)\, d\sigma(x) \geq \lambda \int_\Omega u(\varphi - u)\, dx\,, \tag{2.25}$$

for all $\varphi \in \mathcal{C}$. It is obvious that the *classical* eigenvalues of problem (2.11) (given by relation (2.12)) are also extended eigenvalues. On the other hand, it is also clear that $\mu_1(\alpha)$ is an extended eigenvalue of problem (2.11), for any $\alpha > 0$. Thus, relation (2.25) gives a connection between $\lambda_1(\alpha)$ and $\mu_1(\alpha)$. In fact, if $u \in \mathcal{C} \setminus \{0\}$ is an extended eigenfunction corresponding to some extended eigenvalue $\lambda > 0$ of problem (2.11), then either u is an interior point of $\mathcal{C}$ (i.e., $u = u_1 + c$, for some $u_1 \in V$ and $c > 0$) so that λ is a classical eigenvalue, or $u \in V \setminus \{0\}$ and $v = u$ satisfies (2.24).

It is also worth pointing out the fact that since problem (2.11) has a nonlinear boundary condition, the study of the existence of other eigenvalues (different from λ_0 and $\lambda_1(\alpha)$) is more difficult than in the case of problems involving linear boundary conditions. Methods which are usually used fail in this case. In this context, we just notice that we cannot apply the Ljusternik-Schnirelman theory in this case, since the Euler-Lagrange energetic functional associated with problem (2.11) is not *even*, a crucial condition required by the application of the quoted method. However, in the one-dimensional case the existence of infinitely many eigenvalues can be easily stated. Note that problem (2.11) with $\Omega = (0,1)$ becomes

$$\begin{cases} -u''(t) = \lambda u(t) & \text{for } t \in (0,1)\,, \\ u'(0) = \alpha u_+(0), \quad -u'(1) = \alpha u_+(1)\,. \end{cases} \tag{2.26}$$

On the other hand, it is known (see, e.g., [40, p. 10]) that the one-dimensional Neumann problem

$$\begin{cases} -u''(t) = \lambda u(t) & \text{for } t \in (0,1)\,, \\ u'(0) = u'(1) = 0\,, \end{cases} \tag{2.27}$$

has the eigenvalues $\mu_k = k^2\pi^2$, $k = 0, 1, ...$, with the corresponding eigenfunctions $u_k(t) = -\cos(k\pi t)$. Simple computations show that for each $k \in \mathbf{Z}_+$, μ_{2k} is an eigenvalue of problem (2.26) with the corresponding eigenfunction u_{2k}.

Finally, let us point out that all the discussion on problem (2.11) presented above can be extended (by using similar arguments) to the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \ \Omega, \\ -|\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \alpha u_+^{p-1} & \text{on } \partial\Omega\,, \end{cases}$$

where $p \in (1, N)$ is a real number and $\Delta_p \cdot = \operatorname{div}(|\nabla \cdot|^{p-2}\nabla \cdot)$ stands for the p-Laplace operator.

2.3 An eigenvalue problem for the Laplace operator with Neumann boundary condition

2.3.1 Introduction and main results

Assume $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with smooth boundary $\partial\Omega$. Denote by ν the outward unit normal to $\partial\Omega$. A classical result in the theory of eigenvalue problems assures that problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.28}$$

possesses a sequence of non-negative eigenvalues (going to $+\infty$) and a sequence of corresponding eigenfunctions which define a Hilbert basis in $L^2(\Omega)$ (see, e.g. [40, Theorem 1.2.8]). Moreover, it is known that the first eigenvalue of problem (2.28) is $\lambda = 0$ and it is isolated and simple (see, e.g. [34, Proposition 4.2.1]). Furthermore, the second eigenvalue is characterized from a variational point of view in the following way

$$\lambda_1^N := \inf_{u \in W^{1,2}(\Omega)\setminus\{0\},\ \int_\Omega u\,dx = 0} \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\displaystyle\int_\Omega u^2\,dx}\,.$$

Assume that $p > 2$ is a given real number and consider the eigenvalue problem

$$\begin{cases} -\Delta_p\, u = \lambda u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.29}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ stands for the p-Laplace operator. Using a variational technique based on the fact that the energy functional associated to this problem has a nontrivial minimum for any positive λ it is easy to show that the set of eigenvalues of problem (2.29) is exactly the interval $[0, \infty)$. In other words, the set of eigenvalues in this case is a continuous family.

In this section we consider it is important to point out a new situation which can occur in the study of eigenvalue problems for elliptic operators involving homogeneous Neumann boundary conditions. More exactly, we analyze the following eigenvalue problem

$$\begin{cases} -\Delta_p u - \Delta u = \lambda u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.30}$$

where $\lambda \in \mathbb{R}$ and $p > 2$ is a real number. We will show that this problem possesses, on the one hand, a continuous family of eigenvalues and, on the other hand, exactly one more eigenvalue, which is isolated in the set of eigenvalues of problem (2.30). Since $p > 2$ (and consequently $W^{1,p}(\Omega) \subset W^{1,2}(\Omega)$) it is natural to analyze equation (2.30) in the Sobolev space $W^{1,p}(\Omega)$. Consequently, we will say that $\lambda \in \mathbb{R}$

is an *eigenvalue* of problem (2.30) if there exists $u_\lambda \in W^{1,p}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega (|\nabla u_\lambda|^{p-2} + 1)\nabla u_\lambda \nabla \varphi \, dx - \lambda \int_\Omega u_\lambda \varphi \, dx = 0 \,, \tag{2.31}$$

for all $\varphi \in W^{1,p}(\Omega)$. Such a function u_λ will be called an *eigenfunction* corresponding to eigenvalue λ.

The first main result of this section is given by the following theorem.

Theorem 2.3. *For each $p > 2$ define*

$$\lambda_1(p) := \inf_{u \in W^{1,p}(\Omega)\setminus\{0\},\ \int_\Omega u\, dx = 0} \frac{\frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx}{\frac{1}{2}\int_\Omega u^2 \, dx} \,. \tag{2.32}$$

Then $\lambda_1(p) > 0$ and for each $p > 2$ fixed, the set of eigenvalues of problem (2.30) is given by

$$\{0\} \cup (\lambda_1(p), \infty) \,.$$

We point out the fact that a similar result with the one of Theorem 2.3 was obtained in [55] for a problem of type (2.30) with a homogeneous Dirichlet boundary condition instead of the homogeneous Neumann boundary condition considered in this section. However, in [55] only the existence of a continuous family of eigenvalues was established. Thus, the result of Theorem 2.3 here is more interesting in this new case. Furthermore, Theorem 2.3 here describes completely the set of eigenvalues of problem (2.30) while the result in [55] does not describe the entire set of eigenvalues of the problem studied there. The existence of a continuous family of eigenvalues for problem (2.30) is a direct consequence of the fact that we deal with a *non-homogeneous* eigenvalue problem while the presence of the isolated eigenvalue $\lambda_0 = 0$ is a consequence of the *boundary condition* considered in relation to problem (2.30). On the other hand, we notice that the proofs in this new situation ask for a different framework than the one used in [55] since we deal with another type of boundary condition.

Finally, we recall that results concerning a continuous family of eigenvalues plus one more isolated point were also obtained for a different eigenvalue problem involving a homogeneous Dirichlet boundary condition in [60].

Next, we define function $\lambda_1 : [2, \infty) \to [\lambda_1^N, \infty)$ where $\lambda_1(p)$ is given by expression (2.32) from Theorem 2.3 if $p \neq 2$ and $\lambda_1(2) = 2\lambda_1^N$, where λ_1^N stands for the second eigenvalue of problem (2.28). Our second main result presents certain properties of function λ_1 defined above.

Theorem 2.4. *a) Function $\lambda : (2, \infty) \to [\lambda_1^N, \infty)$ is non-decreasing.*

b) For each $p \in (2, \infty)$ we have

$$\lim_{s \nearrow p} \lambda_1(s) \leq \lambda_1(p) \leq \lim_{s \searrow p} \lambda_1(s) \,.$$

c) Function $\lambda_1 : [2, \infty) \to [\lambda_1^N, \infty)$ *is bounded from above.*

d) If $\lambda_1^N \geq 2$ *then there exists* $p_0 \geq 2$ *such that*

$$\lambda_1(p_0) = p_0 \,.$$

Remark 2.4. *We note that hypotheses* $\lambda_1^N \geq 2$ *can occur. For instance if* Ω *is the ball of radius* 1 *and centered in the origin in* $\mathbb{R}^N$ *then* $\lambda_1^N \geq \pi^2 > 2$ *(see, e.g. [40, Chapter 7, p.101] or L. Payne and H. Weinberger [68]).*

Remark 2.5. *By Theorems 2.3 and 2.4 we deduce that there exists* $p > 2$ *for which the set of eigenvalues of problem (2.30) is given by*

$$\{0\} \cup (p, \infty) \,.$$

2.3.2 Proof of Theorem 2.3

Let $p > 2$ be arbitrary but fixed. The proof of Theorem 2.3 will follow as a direct consequence of the lemmas proved in this section.

Lemma 2.15. *a)* $\lambda_0 = 0$ *is an eigenvalue of problem (2.30).*

b) Any $\lambda < 0$ *is not an eigenvalue of problem (2.30).*

Proof. a) The fact that $\lambda_0 = 0$ is an eigenvalue of problem (2.30) is obvious since it verifies relation (2.31) with u_0 equal to any real constant.

b) Assume that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (2.30) with $u_\lambda \in W^{1,p}(\Omega) \setminus \{0\}$ the corresponding eigenfunction. Then, relation (2.31) with $\varphi = u_\lambda$ implies

$$\lambda \int_\Omega u_\lambda^2 \, dx = \int_\Omega (|\nabla u_\lambda|^p + |\nabla u_\lambda|^2) \, dx \geq 0 \,.$$

Consequently, we obtain $\lambda \geq 0$ and, thus, any $\lambda < 0$ can not be an eigenvalue of problem (2.30).

Remark 2.6. *In order to go further, let us remember that for each* $p > 1$ *we can define a (closed) subspace of* $W^{1,p}(\Omega)$ *by*

$$V_p := \{u \in W^{1,p}(\Omega); \ \int_\Omega u \, dx = 0\} \,.$$

It is well-known that

$$W^{1,p}(\Omega) = V_p \oplus \mathbb{R} \,,$$

and the Poincaré-Wirtinger inequality holds true (see, e.g. [13, p. 194]), i.e. there exists a positive constant C_p *such that*

$$\int_\Omega |u|^p \, dx \leq C_p \int_\Omega |\nabla u|^p \, dx, \quad \forall \, u \in V_p \,. \tag{2.33}$$

Moreover, if $1 < q < p$ *then* $V_p \subset V_q$*.*

The above discussion shows that the positive eigenvalues of problem (2.30) should have the corresponding eigenfunctions in V_p. On the other hand, using the notations introduced in Remark 2.6 we have that number $\lambda_1(p)$ defined in Theorem 2.3 can be characterized by the following relation

$$\lambda_1(p)=\inf_{u\in V_p\setminus\{0\}}\frac{\dfrac{1}{p}\displaystyle\int_\Omega|\nabla u|^p\,dx+\dfrac{1}{2}\int_\Omega|\nabla u|^2\,dx}{\dfrac{1}{2}\displaystyle\int_\Omega u^2\,dx}\,.$$

A first consequence of Remark 2.6 is the result of the following lemma.

Lemma 2.16. $\lambda_1(p)>0$.

Proof. Since $2<p$ we deduce by Remark 2.6 that $V_p\subset V_2$. Thus, relation (2.33) with $p=2$ yields

$$\int_\Omega u^2\,dx\le C_2\int_\Omega|\nabla u|^2\,dx,\quad\forall\,u\in V_p\subset V_2\,.$$

Consequently, we find

$$\frac{1}{p}\int_\Omega|\nabla u|^p\,dx+\frac{1}{2}\int_\Omega|\nabla u|^2\,dx\ge\frac{1}{2C_2}\int_\Omega u^2\,dx,\quad\forall\,u\in V_p\,,$$

or

$$\lambda_1(p)\ge\frac{2}{C_2p}>0\,.$$

A careful analysis of all the above ideas shows that in order to prove that every $\lambda\in[\lambda_1(p),\infty)$ is an eigenvalue of problem (2.30), it is enough to solve equation (2.30) in V_p (instead of $W^{1,p}(\Omega)$). That fact is mainly due to the remark that $W^{1,p}(\Omega)=V_p\oplus\mathbb{R}$.

Lemma 2.17. *For each $\lambda>0$ we have*

$$\lim_{\|u\|_{W^{1,p}(\Omega)}\to\infty,\;u\in V_p}\left(\frac{1}{p}\int_\Omega|\nabla u|^p\,dx+\frac{1}{2}\int_\Omega|\nabla u|^2\,dx-\frac{\lambda}{2}\int_\Omega u^2\,dx\right)=\infty\,,$$

for each $\lambda>0$.

Proof. Let $\lambda>0$ be arbitrary but fixed. Relation (2.33) yields

$$\begin{aligned}\frac{1}{p}\int_\Omega|\nabla u|^p\,dx+\frac{1}{2}\int_\Omega|\nabla u|^2\,dx&\ge\frac{1}{2p}\int_\Omega|\nabla u|^p\,dx+\frac{1}{2pC_p}\int_\Omega|u|^p\,dx\\&\ge C\left(\int_\Omega|\nabla u|^p\,dx+\int_\Omega|u|^p\,dx\right),\quad\forall\,u\in V_p\,,\end{aligned}$$

where $C=\frac{1}{2p}\min\{1,1/C_p\}>0$ is a constant. The last inequality can be also written as

$$\frac{1}{p}\int_\Omega|\nabla u|^p\,dx+\frac{1}{2}\int_\Omega|\nabla u|^2\,dx\ge C\|u\|^p_{W^{1,p}(\Omega)},\quad\forall\,u\in V_p\,.\tag{2.34}$$

Next, relation (2.33) with $p = 2$ implies the existence of a positive constant C_2 such that

$$\int_\Omega u^2\,dx \le C_2 \int_\Omega |\nabla u|^2\,dx, \quad \forall\, u \in V_2\,. \tag{2.35}$$

Since $2 < p$ we can use Remark 2.6 in order to obtain $V_p \subset V_2$. Thus, inequality (2.35) holds true for any $u \in V_p$. On the other hand, $u \in V_p$ implies $|\nabla u| \in L^p(\Omega)$. Applying Hölder's inequality we obtain

$$\int_\Omega |\nabla u|^2\,dx \le |\Omega|^{(p-2)/p} \left(\int_\Omega |\nabla u|^p\,dx\right)^{2/p} \le |\Omega|^{(p-2)/p} \|u\|^2_{W^{1,p}(\Omega)}, \quad \forall\, u \in V_p\,. \tag{2.36}$$

By inequalities (2.35) and (2.36) we get

$$\int_\Omega u^2\,dx \le D\|u\|^2_{W^{1,p}(\Omega)}, \quad \forall\, u \in V_p\,, \tag{2.37}$$

where $D > 0$ is a constant. Finally, we notice that relations (2.34) and (2.37) lead to the following inequality

$$\frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{2}\int_\Omega u^2\,dx \ge C\|u\|^p_{W^{1,p}(\Omega)} - \lambda D\|u\|^2_{W^{1,p}(\Omega)}, \quad \forall\, u \in V_p\,.$$

The last inequality and $p > 2$ show that the conclusion of Lemma 2.17 holds true.

Lemma 2.18. *Every $\lambda \in (\lambda_1(p), \infty)$ is an eigenvalue of problem (2.30).*

Proof. For each $\lambda > \lambda_1(p)$ define $T_\lambda : V_p \to \mathbb{R}$ by

$$T_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{2}\int_\Omega u^2\,dx, \quad \forall\, u \in V_p\,.$$

Standard arguments show that $T_\lambda \in C^1(V_p, \mathbb{R})$ with the derivative given by

$$\langle T_\lambda'(u), \varphi\rangle = \int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{2-2})\nabla u \nabla\varphi\,dx - \lambda\int_\Omega u\varphi\,dx\,,$$

for all u, $\varphi \in V_p$.

Lemma 2.17 shows that T_λ is coercive in V_p, i.e.

$$\lim_{\|u\|_{W^{1,p}(\Omega)}\to\infty,\; u\in V_p} T_\lambda(u) = \infty\,.$$

On the other hand, T_λ is also inferior weakly semicontinuous on V_p. Then we can apply [75, Theorem 1.2] in order to obtain the existence of a global minimum point of T_λ, denoted by $\theta_\lambda \in V_p$, i.e. $T_\lambda(\theta_\lambda) = \min_{V_p} T_\lambda$. Using the definition of $\lambda_1(p)$ we deduce that for any $\lambda > \lambda_1(p)$ there exists $w_\lambda \in V_p$ such that

$$T_\lambda(w_\lambda) < 0\,,$$

or

$$T_\lambda(\theta_\lambda) \le T_\lambda(w_\lambda) < 0\,,$$

that means, $\theta_\lambda \neq 0$. In other words, $\theta_\lambda \in V_p \setminus \{0\} \subset W^{1,p}(\Omega) \setminus \{0\}$. On the other hand, standard arguments show that the following relation is satisfied

$$\langle T_\lambda^{'}(\theta_\lambda), \varphi \rangle = 0, \quad \forall\, \varphi \in V_p\,.$$

But the above equality also holds true if φ is a real constant (since $\theta_\lambda \in V_p$ and, thus, $\int_\Omega \theta_\lambda \, dx = 0$). Taking into account the fact that by Remark 2.6 we have $W^{1,p}(\Omega) = V_p \oplus \mathbb{R}$ we find that

$$\langle T_\lambda^{'}(\theta_\lambda), \varphi \rangle = 0, \quad \forall\, \varphi \in W^{1,p}(\Omega)\,,$$

with $\theta_\lambda \in V_p \setminus \{0\} \subset W^{1,p}(\Omega) \setminus \{0\}$. Consequently, each $\lambda > \lambda_1(p)$ is an eigenvalue of problem (2.30).

Lemma 2.19. *Each $\lambda \in (0, \lambda_1(p))$ is not an eigenvalue of problem (2.30).*

Proof. Indeed, assuming by contradiction that $\lambda \in (0, \lambda_1(p))$ is an eigenvalue of (2.30) with $u_\lambda \in V_p \setminus \{0\}$ the corresponding eigenfunction by the definition of $\lambda_1(p)$ and relation (2.31) with $\varphi = u_\lambda$ we get

$$\begin{aligned} 0 < \frac{\lambda_1(p) - \lambda}{2} \int_\Omega u_\lambda^2 \, dx &\le \frac{1}{p} \int_\Omega |\nabla u_\lambda|^p \, dx + \frac{1}{2} \int_\Omega |\nabla u_\lambda|^2 \, dx - \frac{\lambda}{2} \int_\Omega u_\lambda^2 \, dx \\ &\le \frac{1}{2} \int_\Omega |\nabla u_\lambda|^p \, dx + \frac{1}{2} \int_\Omega |\nabla u_\lambda|^2 \, dx - \frac{\lambda}{2} \int_\Omega u_\lambda^2 \, dx \\ &= 0\,. \end{aligned}$$

Clearly, we obtained a contradiction which shows that the conclusion of Lemma 2.19 is valid.

Lemma 2.20. *Number $\lambda_1(p)$ defined in Theorem 2.3 is not an eigenvalue of problem (2.30).*

Proof. In order to prove this lemma let us first define the quantity

$$\nu_1^N(p) := \inf_{u \in V_p \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{\displaystyle\int_\Omega u^2 \, dx}\,.$$

Note that $\nu_1^N(p)$ is not the second eigenvalue of problem (2.28), namely λ_1^N, since in the above expression the infimum is taken after $u \in V_p \setminus \{0\}$ and not after $u \in V_2 \setminus \{0\}$. Since $V_p \subset V_2$, we actually have

$$\nu_1^N(p) \ge \lambda_1^N > 0\,.$$

Next, notice that for each $u \in V_p \setminus \{0\}$ and each $t > 0$ real number we have

$$\begin{aligned} \lambda_1(p) &\le \frac{\displaystyle\frac{1}{p} \int_\Omega |\nabla (tu)|^p \, dx + \frac{1}{2} \int_\Omega |\nabla (tu)|^2 \, dx}{\displaystyle\frac{1}{2} \int_\Omega (tu)^2 \, dx} \\ &= \frac{2t^{p-2}}{p} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega u^2 \, dx} + \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{\displaystyle\int_\Omega u^2 \, dx}\,. \end{aligned}$$

Thus, for each $u \in V_p \setminus \{0\}$ fixed passing to the limit as $t \to 0$ we find

$$\lambda_1(p) \leq \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx}{\displaystyle\int_\Omega u^2\, dx}\,.$$

Since the above inequality holds true for each $u \in V_p \setminus \{0\}$, passing to the infimum in the right-hand side when $u \in V_p \setminus \{0\}$ we get

$$\lambda_1(p) \leq \nu_1^N(p)\,.$$

On the other hand, for each $u \in V_p \setminus \{0\}$, arbitrary but fixed, we have

$$\frac{\displaystyle\frac{1}{p}\int_\Omega |\nabla u|^p\, dx + \frac{1}{2}\int_\Omega |\nabla u|^2\, dx}{\displaystyle\frac{1}{2}\int_\Omega u^2\, dx} \geq \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx}{\displaystyle\int_\Omega u^2\, dx} \geq \nu_1^N(p)\,.$$

Passing to the infimum in the left-hand side when $u \in V_p \setminus \{0\}$ we get

$$\lambda_1(p) \geq \nu_1^N(p)\,.$$

Consequently, we find that

$$\lambda_1(p) = \nu_1^N(p)\,.$$

Finally, let us assume by contradiction that $\lambda_1(p) = \nu_1^N(p)$ is an eigenvalue of problem (2.30) with $u_\lambda \in V_p \setminus \{0\}$ the corresponding eigenfunction. By the definition of $\lambda_1(p)$ and relation (2.31) with $\varphi = u_\lambda$ we obtain

$$\int_\Omega |\nabla u_\lambda|^p\, dx + \nu_1^N(p)\int_\Omega u_\lambda^2\, dx \leq \int_\Omega |\nabla u_\lambda|^p\, dx + \int_\Omega |\nabla u_\lambda|^2\, dx = \lambda_1(p)\int_\Omega u_\lambda^2\, dx\,.$$

It follows that

$$\int_\Omega |\nabla u_\lambda|^p\, dx = 0\,,$$

and combining this with relation (2.33) we deduce that $u_\lambda = 0$, a contradiction.

The proof of Lemma 2.20 is complete.

2.3.3 Proof of Theorem 2.4

a) Following the first part of the proof of Lemma 2.20 we deduce that for each $p > 2$ we can characterize $\lambda_1(p)$ as the following infimum

$$\lambda_1(p) := \inf_{u \in V_p \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx}{\displaystyle\int_\Omega u^2\, dx}\,.$$

Thus, it is obvious that for each $p > 1$ we have $\lambda_1(p) \geq \lambda_1^N$ and since for each $p > q > 2$ we have $V_p \subset V_q$ we deduce that

$$\lambda_1(p) \geq \lambda_1(q), \quad \forall\, p > q > 2\,,$$

in other words, λ_1 is non-decreasing on $(2, \infty)$.

b) Since by a) λ_1 is monotone (non-decreasing) it follows that it has one-sided limits.

c) Let us introduce the distance function $d : \Omega \to \mathbb{R}$, by

$$d(x) = \mathrm{dist}(x, \partial\Omega)\,,$$

for all $x \in \Omega$. It is easy to observe that d is Lipschitz continuous and satisfies

$$|\nabla d(x)| = 1, \quad \text{for a.e. } x \in \Omega\,.$$

Next, define $\psi : \Omega \to \mathbb{R}$ by

$$\psi(x) = d(x) - \frac{1}{|\Omega|}\int_\Omega d(y)\, dy\,.$$

Obviously, $\psi \in V_p$, for any $p > 2$ and $\int_\Omega \psi^2\, dx > 0$. That facts and relation (2.32) yield

$$\begin{aligned}\lambda_1(p) &\leq \frac{\dfrac{1}{p}\displaystyle\int_\Omega |\nabla\psi|^p\, dx + \dfrac{1}{2}\displaystyle\int_\Omega |\nabla\psi|^2\, dx}{\dfrac{1}{2}\displaystyle\int_\Omega \psi^2\, dx}\\ &= \frac{\dfrac{1}{p}\displaystyle\int_\Omega |\nabla d|^p\, dx + \dfrac{1}{2}\displaystyle\int_\Omega |\nabla d|^2\, dx}{\dfrac{1}{2}\displaystyle\int_\Omega \psi^2\, dx} = \frac{2|\Omega|}{p\displaystyle\int_\Omega \psi^2\, dx}\,,\end{aligned}$$

for every $p > 2$. Consequently, function $\lambda_1(p)$ is bounded from above by the constant

$$\Lambda = \max\left\{\frac{2|\Omega|}{p\displaystyle\int_\Omega \psi^2\, dx}, 2\lambda_1^N\right\}\,, \tag{2.38}$$

for each $p \geq 2$.

d) By a) and c) we deduce that there exists $\Lambda_1 \in (\lambda_1^N, \Lambda]$ such that

$$\lim_{p\to\infty} \lambda_1(p) = \sup_{p\geq 2} \lambda_1(p) = \Lambda_1\,,$$

where Λ is given by relation (2.38). Next, we deduce that λ_1 is a non-decreasing function satisfying $\lambda_1([2, \Lambda_1]) \subset [\lambda_1^N, \Lambda_1] \subset [2, \Lambda_1]$. An elementary result in mathematical analysis asserts that such a function possesses a fixed point.

Theorem 2.4 is completely proved.

Chapter 3

Dirichlet eigenvalue problems involving variable exponent growth conditions

3.1 Eigenvalue problem $-\Delta_{p(x)}u = \lambda|u|^{p(x)-2}u$

Elliptic equations involving variable exponent growth conditions have been intensively discussed in the last decade. A strong motivation in studying such kind of problems is due to the fact that they can model with high accuracy various phenomena which arise from the study of elastic mechanics (see, V. Zhikov [81]), electrorheological fluids (see, E. Acerbi and G. Mingione [1, 2], L. Diening [20], T. C. Halsey [38], M. Ruzicka [72, 73]) or image restoration (see, Y. Chen, S. Levine and R. Rao [17]). In that context, eigenvalue problems involving variable exponent growth conditions represent a starting point in analyzing more complicated equations. A first contribution in this sense is the paper of X. Fan, Q. Zhang and D. Zhao [30] where the following eigenvalue problem has been considered

$$\begin{cases} -\Delta_{p(x)}u = \lambda|u|^{p(x)-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p : \overline{\Omega} \to (1, \infty)$ is a continuous function, $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ stands for the $p(x)$-Laplace operator and λ is a real number. The result in [30] establishes the existence of infinitely many eigenvalues for problem (3.1) by using an argument based on the Ljusternik-Schnirelmann critical point theory. Denoting by Λ the set of all nonnegative eigenvalues, the authors showed that $\sup\Lambda = +\infty$ and they pointed out that only under special conditions, which are somehow connected with a kind of monotony of function $p(x)$, we have $\inf\Lambda > 0$ (this is in contrast with the case when $p(x)$ is a constant; then, we always have $\inf\Lambda > 0$).

We notice that the above discussion is in keeping with the fact that considering, the Rayleigh quotient associated with problem (3.1), that is

$$\mu_1 := \inf_{u \in C_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^{p(x)}\, dx}{\displaystyle\int_\Omega |u|^{p(x)}\, dx}\,,$$

we often have $\mu_1 = 0$ for general $p(x)$. An example in that sense is illustrated by X. Fan and D. Zhao in [31], pages 444-445. More exactly, letting $\Omega = (-2, 2) \subset \mathbf{R}$ and defining $p(x) = 3$ if $0 \leq |x| \leq 1$, and $p(x) = 4 - |x|$ if $1 \leq |x| \leq 2$ it can be proved that $\mu_1 = 0$ (see also [30, Theorem 3.1] for a more general result when $\mu_1 = 0$). On the other hand, a necessary and sufficient condition such that $\mu_1 > 0$ has not yet been obtained excepting the case when $N = 1$ (in that case, the infimum is positive if and only if $p(x)$ is a monotone function, see [30, Theorem 3.2]). However, the authors of [30] pointed out that in the case $N > 1$ a sufficient condition to have $\mu_1 > 0$ is to exist a vector $l \in \mathbf{R}^N \setminus \{0\}$ such that, for any $x \in \Omega$, function $f(t) = p(x + tl)$ is monotone, for $t \in I_x := \{s;\ x + sl \in \Omega\}$ (see [30, Theorem 3.3]). Assuming p is of class C^1 the monotony of function f reads as follows: either

$$\nabla p(x + tl) \cdot l \geq 0, \quad \text{for all } t \in I_x,\ x \in \Omega\,,$$

or

$$\nabla p(x + tl) \cdot l \leq 0, \quad \text{for all } t \in I_x,\ x \in \Omega\,.$$

We can supplement the above results in a sense that will be described below.

Assume $\Omega \subset \mathbf{R}^N$ $(N \geq 2)$ is an open, bounded and smooth set. For each $x \in \Omega$, $x = (x_1, ..., x_N)$ and $i \in \{1, ..., N\}$ we denote

$$m_i = \inf_{x \in \Omega} x_i \qquad M_i = \sup_{x \in \Omega} x_i\,.$$

For each $i \in \{1, ..., N\}$ let $a_i : [m_i, M_i] \to \mathbf{R}$ be functions of class C^1. Particularly, functions a_i are allowed to vanish.

Let $\overrightarrow{a} : \Omega \to \mathbf{R}^N$ be defined by

$$\overrightarrow{a}(x) = (a_1(x_1), ..., a_N(x_N))\,.$$

We assume that there exists $a_0 > 0$ a constant such that

$$\operatorname{div} \overrightarrow{a}(x) \geq a_0 > 0, \quad \forall\, x \in \overline{\Omega}\,. \tag{3.2}$$

Next, we consider $p : \overline{\Omega} \to (1, N)$ is a function of class C^1 satisfying

$$\overrightarrow{a}(x) \cdot \nabla p(x) = 0, \quad \forall\, x \in \Omega\,. \tag{3.3}$$

We point out the following result which can be found in [63].

Theorem 3.1. *Assume that $\overrightarrow{a}(x)$ and $p(x)$ are defined as above and satisfy conditions (3.2) and (3.3). Then there exists a positive constant C such that*

$$\int_\Omega |u(x)|^{p(x)}\, dx \leq C \int_\Omega |\overrightarrow{a}(x)|^{p(x)} |\nabla u(x)|^{p(x)}\, dx, \quad \forall\, u \in C_0^1(\Omega)\,. \tag{3.4}$$

Proof. The proof of Theorem 3.1 is inspired by the ideas in [78, Théorème 20.7].

Simple computations based on relation (3.3) show that for each $u \in C_0^1(\Omega)$ the following equality holds true

$$\begin{aligned}\operatorname{div}(|u(x)|^{p(x)}\overrightarrow{a}(x)) &= \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(|u(x)|^{p(x)}a_i(x_i)\right)\\ &= |u(x)|^{p(x)}\operatorname{div}\overrightarrow{a}(x)\\ &+ \sum_{i=1}^{N}a_i(x_i)\left[p(x)|u(x)|^{p(x)-2}u(x)\frac{\partial u}{\partial x_i}+|u(x)|^{p(x)}\log(|u(x)|)\frac{\partial p}{\partial x_i}\right]\\ &= |u(x)|^{p(x)}\operatorname{div}\overrightarrow{a}(x)+p(x)|u(x)|^{p(x)-2}u(x)\nabla u(x)\cdot\overrightarrow{a}(x)+\\ &\quad |u(x)|^{p(x)}\log(|u(x)|)\nabla p(x)\cdot\overrightarrow{a}(x)\\ &= |u(x)|^{p(x)}\operatorname{div}\overrightarrow{a}(x)+p(x)|u(x)|^{p(x)-2}u(x)\nabla u(x)\cdot\overrightarrow{a}(x)\,.\end{aligned}$$

On the other hand, the flux-divergence theorem implies that for each $u \in C_0^1(\Omega)$ we have

$$\int_\Omega \operatorname{div}(|u(x)|^{p(x)}\overrightarrow{a}(x))\;dx=\int_{\partial\Omega}|u(x)|^{p(x)}\overrightarrow{a}(x)\cdot\overrightarrow{n}\;d\sigma(x)=0\,.$$

Using the above pieces of information we infer that for each $u \in C_0^1(\Omega)$ the following holds true

$$\int_\Omega |u(x)|^{p(x)}\operatorname{div}\overrightarrow{a}(x)\;dx\le p^+\int_\Omega |u(x)|^{p(x)-1}|\nabla u(x)||\overrightarrow{a}(x)|\;dx\,.$$

Next, we recall that for each $\epsilon > 0$, for each $x \in \Omega$ and for each A, $B \geq 0$ the following Young type inequality holds true (see, e.g. [13, the footnote on p. 56])

$$AB\le\epsilon A^{\frac{p(x)}{p(x)-1}}+\frac{1}{\epsilon^{p(x)-1}}B^{p(x)}\,.$$

We fix $\epsilon > 0$ such that

$$p^+\epsilon < a_0\,,$$

where a_0 is given by relation (3.2).

The above facts and relation (3.2) yield

$$a_0\int_\Omega |u(x)|^{p(x)}\;dx\le p^+\left[\epsilon\int_\Omega |u(x)|^{p(x)}\;dx+\int_\Omega\left(\frac{1}{\epsilon}\right)^{p(x)-1}|\overrightarrow{a}(x)|^{p(x)}|\nabla u(x)|^{p(x)}\;dx\right]\,,$$

for any $u \in C_0^1(\Omega)$, or

$$(a_0-\epsilon p^+)\int_\Omega |u(x)|^{p(x)}\;dx\le\left[\left(\frac{1}{\epsilon}\right)^{p^--1}+\left(\frac{1}{\epsilon}\right)^{p^+-1}\right]p^+\int_\Omega|\overrightarrow{a}(x)|^{p(x)}|\nabla u(x)|^{p(x)}\;dx\,,$$

for any $u \in C_0^1(\Omega)$. The conclusion of Theorem 3.1 is now clear.

Example 1. We point out an example of functions $\overrightarrow{a}(x)$ and $p(x)$ satisfying conditions (3.2) and (3.3) in the case when $\overrightarrow{a}(x)$ can vanish in some points of Ω. Let $N \geq 3$ and $\Omega = B_{\frac{1}{\sqrt{N}}}(0)$, the ball centered in the origin of radius $\frac{1}{\sqrt{N}}$. We define $\overrightarrow{a}(x) : \Omega \to \mathbf{R}^N$ by

$$\overrightarrow{a}(x)=(-x_1,x_2,x_3,...,x_{N-1},x_N)\,,$$

(more exactly, function $\overrightarrow{a}(x)$ is associated to a vector $x \in \Omega$ the vector obtained from x by changing in the first position x_1 by $-x_1$ and keeping unchanged x_i for $i \in \{2, ..., N\}$). Clearly, $\overrightarrow{a}(x)$ is of class C^1, $\overrightarrow{a}(0) = 0$ and we have

$$\text{div}(\overrightarrow{a}(x)) = N - 2 \geq 1, \quad \forall\, x \in \Omega\,.$$

Thus, condition (3.2) is satisfied.

Next, we define $p : \overline{\Omega} \to (1, N)$ by

$$p(x) = x_1(x_2 + x_3 + ... + x_{N-1} + x_N) + 2, \quad \forall\, x \in \overline{\Omega}\,.$$

It is easy to check that p is of class C^1 and some elementary computations show that

$$\nabla p(x) \cdot \overrightarrow{a}(x) = (x_2 + ... + x_N)(-x_1) + x_1x_2 + ... + x_1x_N = 0, \quad \forall\, x \in \Omega\,.$$

It means that condition (3.3) is satisfied, too.

Example 2. We point out a second example, for $N = 2$. Taking $\Omega = B_{\frac{1}{3^{1/3}}}(0)$, $\overrightarrow{a}(x) = (-x_1, 2x_2)$ and $p(x) = x_1^2 x_2 + \frac{3}{2}$ it is easy to check that relations (3.2) and (3.3) are fulfilled.

Remark. If N, a and p are as in Example 1 or Example 2 then the result of Theorem 3.1 reads as follows: there exists a positive constant $C > 0$ such that

$$\int_\Omega |u(x)|^{p(x)}\, dx \leq C \int_\Omega |x|^{p(x)} |\nabla u(x)|^{p(x)}\, dx, \quad \forall\, u \in C_0^1(\Omega)\,. \tag{3.5}$$

3.2 Eigenvalue problem $-\Delta_{p(x)} u = \lambda |u|^{q(x)-2} u$

Going further, another eigenvalue problem involving variable exponent growth conditions intensively studied is the following

$$\begin{cases} -\Delta_{p(x)} u = \lambda |u|^{q(x)-2} u & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.6}$$

where $\Omega \subset \mathbf{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$, p, $q : \overline{\Omega} \to (1, \infty)$ are two continuous functions and λ is a real number. In the case when $p(x) \neq q(x)$ the competition between the growth rates involved in equation (3.6) is essential in describing the set of eigenvalues of this problem. Thus, in the case when $\min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x)$ and $q(x)$ has a subcritical growth Ekeland's variational principle can be used (see the paper, M. Mihăilescu and V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proceedings Amer. Math. Soc.* **135** (2007) 2929-2937.) in order to prove the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin. This result was later extended by X. Fan in [27]. In the case when $\max_{x\in\overline{\Omega}} p(x) < \min_{x\in\overline{\Omega}} q(x)$ and $q(x)$ has a subcritical growth, a mountain-pass argument, similar with that used by Fan and Zhang [29], can be applied in order to show that any $\lambda > 0$ is an eigenvalue of problem (3.6). Finally, in the case when $\max_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x)$ it can be proved that the energetic functional which can be associated with the eigenvalue problem has a nontrivial minimum for any $\lambda > 0$ (see, [29]).

For more information and connections regarding the study of eigenvalue problems involving variable exponent growth conditions we also refer to [39], (see, the web-site of the *Research group on variable exponent Lebesgue and Sobolev spaces*, `http://www.math.helsinki.fi/analysis/varsobgroup/`).

3.3 An eigenvalue problem involving the $p(x)$-Laplace operator and a non-local term

In this section we point out an eigenvalue problem involving variable exponent growth conditions and a non-local term. With that end in view, let $\Omega \subset \mathbf{R}^N$, $(N \geq 3)$, be a bounded domain with smooth boundary $\partial\Omega$. We analyze the eigenvalue problem

$$\begin{cases} -\eta[u] \cdot \Delta_{p(x)} u = \lambda f(x,u), & \text{for} \quad x \in \Omega \\ u = 0, & \text{for} \quad x \in \partial\Omega\,, \end{cases} \tag{3.7}$$

where $p : \overline{\Omega} \to (1,\infty)$ is a continuous function, $\eta[u]$ is a non-local term defined by the following relation

$$\eta[u] = 2 + \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right)^{\max_{\overline{\Omega}} p / \min_{\overline{\Omega}} p - 1} + \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right)^{\min_{\overline{\Omega}} p / \max_{\overline{\Omega}} p - 1},$$

λ is a real number and $f = f(x,t) : \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ is given by relation

$$f(x,t) := \begin{cases} |t|^{p(x)-2} t, & \text{if} \quad |t| < 1 \\ |t|^{r(x)-2} t, & \text{if} \quad |t| \geq 1\,, \end{cases}$$

with $r : \overline{\Omega} \to (1,\infty)$ a continuous function satisfying

$$\frac{(\max_{\overline{\Omega}} p)^2}{\min_{\overline{\Omega}} p} < \min_{\overline{\Omega}} r \leq \max_{\overline{\Omega}} r < \frac{N \min_{\overline{\Omega}} p}{N - \min_{\overline{\Omega}} p}\,.$$

For problem (3.7) we can prove the existence of a continuous set of eigenvalues in a neighborhood at the right of the origin by using as main argument the mountain-pass theorem. We notice that problem (3.7) is connected with problem (3.1) since near the origin $f(x,t) = |t|^{p(x)-2}t$ and also with problem (3.6) since far from the origin $f(x,t) = |t|^{r(x)-2}t$, with $\min_{\overline{\Omega}} r > \max_{\overline{\Omega}} p$. On the other hand, the presence of the non-local term $\eta[u]$ balances the absence of homogeneity which occurs in the case of variable exponent growth conditions. Particularly, the presence of $\eta[u]$ will help us to formulate a Poincaré type inequality which will be essential in the variational approach considered in order to study problem (3.7) (see Proposition 3.1 below).

We develop the above ideas.

Definition 3.1. *We say $u \in W_0^{1,p(\cdot)}(\Omega)$ is a weak solution for problem (3.7) if*

$$\eta[u] \cdot \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx - \lambda \int_\Omega f(x,u) v \, dx = 0,$$

for all $v \in W_0^{1,p(\cdot)}(\Omega)$. Moreover, we say that $\lambda \in \mathbf{R}$ is an eigenvalue of problem (3.7) if the weak solution u defined above is not trivial.

Define

$$\nu_1 := \inf_{u \in E \setminus \{0\}} \frac{2 \displaystyle\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \frac{p^-}{p^+} \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right)^{p^+/p^-} + \frac{p^+}{p^-} \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right)^{p^-/p^+}}{\displaystyle\int_\Omega \frac{1}{p(x)} |u|^{p(x)} \, dx},$$

where $E = W_0^{1,p(\cdot)}(\Omega)$. A key result regarding ν_1 is given by the following proposition.

Proposition 3.1. *Assume that $p : \overline{\Omega} \to (1, \infty)$ is a continuous function. Then $\nu_1 > 0$.*

Remark. In the particular case when $p(x)$ is a constant function on $\overline{\Omega}$, say $p(x) = p > 1$ for any $x \in \overline{\Omega}$, then $\nu_1 = 4\lambda_1$, where λ_1 is defined by relation

$$\lambda_1 := \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}. \tag{3.8}$$

The main result on problem (3.7) is given by the following theorem.

Theorem 3.2. *Assume $(p^+)^2/p^- < r^- \le r^+ < Np^-/(N - p^-)$. Then any $\lambda \in (0, \nu_1)$ is an eigenvalue of problem (3.7).*

In light of the above remark, we point out the following corollary which represents a particular case of Theorem 3.2 obtained in the case when $p(x) = p > 1$ for any $x \in \overline{\Omega}$, where p is a constant.

Corollary 3.1. *Assume $p(x) = p > 1$ for any $x \in \overline{\Omega}$, where p is a constant, $p < r^- \le r^+ < Np/(N-p)$ and λ_1 is defined by relation (3.8). Then any $\lambda \in (0, 4\lambda_1)$ is an eigenvalue of problem (3.7).*

Let $\lambda \in (0, \nu_1)$ be fixed. The energy functional corresponding to problem (3.7) is defined as $J : W_0^{1,p(\cdot)}(\Omega) \to \mathbf{R}$,

$$\begin{aligned} J(u) &= 2 \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \frac{p^-}{p^+} \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right)^{p^+/p^-} + \\ & \frac{p^+}{p^-} \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \right)^{p^-/p^+} - \lambda \int_\Omega F(x,u) \, dx \end{aligned}$$

where $F(x,u) = \int_0^u f(x,t) \, dt$.

It is known that operator $\Lambda : W_0^{1,p(\cdot)}(\Omega) \to \mathbf{R}$,

$$\Lambda(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx$$

satisfies $\Lambda \in C^1(W_0^{1,p(\cdot)}(\Omega), \mathbf{R})$ with

$$\langle \Lambda^{'}(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v\, dx$$

for all u, $v \in W_0^{1,p(\cdot)}(\Omega)$ (see e.g. [29]).

Defining Λ_1, $\Lambda_2 : W_0^{1,p(\cdot)}(\Omega) \to \mathbf{R}$,

$$\Lambda_1(u) = \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right)^{p^+/p^-} \quad \text{and} \quad \Lambda_2(u) = \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right)^{p^-/p^+}$$

we observe that

$$\Lambda_1(u) = (\Lambda(u))^{p^+/p^-} \quad \text{and} \quad \Lambda_2(u) = (\Lambda(u))^{p^-/p^+}.$$

Thus, it is easy to verify that $\Lambda_1 \in C^1(W_0^{1,p(\cdot)}(\Omega), \mathbf{R})$ and $\Lambda_2 \in C^0(W_0^{1,p(\cdot)}(\Omega), \mathbf{R}) \cap C^1(W_0^{1,p(\cdot)}(\Omega) \setminus \{0\}, \mathbf{R})$ with

$$\langle \Lambda_1^{'}(u), v\rangle = \frac{p^+}{p^-} \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right)^{p^+/p^- -1} \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v\, dx$$

and

$$\langle \Lambda_2^{'}(u), v\rangle = \frac{p^-}{p^+} \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right)^{p^-/p^+ -1} \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v\, dx$$

for all $u \in W_0^{1,p(\cdot)}(\Omega) \setminus \{0\}$, $v \in W_0^{1,p(\cdot)}(\Omega)$.

We deduce that $J \in C^0(W_0^{1,p(\cdot)}(\Omega), \mathbf{R}) \cap C^1(W_0^{1,p(\cdot)}(\Omega) \setminus \{0\}, \mathbf{R})$ with

$$\langle J^{'}(u), v\rangle = \eta[u] \cdot \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v\, dx - \lambda \int_\Omega f(x,u) v\, dx,$$

for all $u \in W_0^{1,p(\cdot)}(\Omega) \setminus \{0\}$, $v \in W_0^{1,p(\cdot)}(\Omega)$. Thus, the weak solutions of (3.7) are exactly the critical points of J. The basic idea to prove Theorem 3.2 is to apply a mountain-pass argument in order to obtain a nontrivial weak solution for problem (3.7), and, thus, to show that $\lambda \in (0, \nu_1)$ is an eigenvalue of (3.7). Here we will present in detail just the result of a lemma which leads to the proof of Proposition 3.1.

Lemma 3.1. *There exists a positive constant $C > 0$ such that the following inequality holds true*

$$\begin{aligned} \int_\Omega |u|^{p(x)}\, dx \quad \leq \quad C \cdot \Bigg[2 \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx + \frac{p^-}{p^+} \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right)^{p^+/p^-} + \\ \frac{p^+}{p^-} \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right)^{p^-/p^+} \Bigg] \end{aligned}$$

for any $u \in W_0^{1,p(\cdot)}(\Omega)$.

Proof. Using relations (1.3) and (1.4) we deduce that for any $u \in W_0^{1,p(\cdot)}(\Omega)$ we have

$$\int_\Omega |u|^{p(x)}\,dx \leq |u|_{p(x)}^{p^+} + |u|_{p(x)}^{p^-}. \tag{3.9}$$

The Sobolev embedding of $W_0^{1,p(\cdot)}(\Omega)$ into $L^{p(\cdot)}(\Omega)$ guarantees the existence of a positive constant $c_1 > 0$ such that

$$|u|_{p(x)} \leq c_1 \|u\|_0 \tag{3.10}$$

for any $u \in W_0^{1,p(\cdot)}(\Omega)$.

Relations (3.9) and (3.10) imply that there exists a positive constant $c_2 > 0$ such that

$$\int_\Omega |u|^{p(x)}\,dx \leq c_2(\|u\|_0^{p^+} + \|u\|_0^{p^-}), \quad \forall\, u \in W_0^{1,p(\cdot)}(\Omega)\,. \tag{3.11}$$

On the other hand, using once again relations (1.3) and (1.4), we find that for any $u \in W_0^{1,p(\cdot)}(\Omega)$

$$\|u\|_0 \leq \left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^{1/p^+} + \left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^{1/p^-}. \tag{3.12}$$

By (3.11) and (3.12) we have

$$\begin{aligned}\int_\Omega |u|^{p(x)}\,dx \quad \leq \quad & c_2 \cdot \left\{\left[\left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^{1/p^+} + \left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^{1/p^-}\right]^{p^+} + \right. \\ & \left.\left[\left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^{1/p^+} + \left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^{1/p^-}\right]^{p^-}\right\}\end{aligned} \tag{3.13}$$

for any $u \in W_0^{1,p(\cdot)}(\Omega)$.

We remember that for any $s > 0$ there exists a positive constant $c_s > 0$ such that

$$(\alpha + \beta)^s \leq c_s(\alpha^s + \beta^s), \quad \forall\ \alpha,\ \beta > 0.$$

Relation (3.13) and the above inequality assure that there exists a positive constant $c_3 > 0$ such that

$$\begin{aligned}\int_\Omega |u|^{p(x)}\,dx \quad \leq \quad & c_3 \cdot \left[2\int_\Omega |\nabla u|^{p(x)}\,dx + \left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^{p^+/p^-} + \right. \\ & \left.\left(\int_\Omega |\nabla u|^{p(x)}\,dx\right)^{p^-/p^+}\right]\end{aligned}$$

for any $u \in W_0^{1,p(\cdot)}(\Omega)$. By the above inequality we conclude that Lemma 3.1 holds true.

3.4 Eigenvalue problem $-\Delta_{p_1(x)}u - \Delta_{p_2(x)}u = \lambda|u|^{q(x)-2}u$

We are concerned with the study of the eigenvalue problem

$$\begin{cases} -\Delta_{p_1(x)}u - \Delta_{p_2(x)}u = \lambda|u|^{q(x)-2}u, & \text{for} \quad x \in \Omega \\ u = 0, & \text{for} \quad x \in \partial\Omega\,, \end{cases} \tag{3.14}$$

where $\Omega \subset \mathbf{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, λ is a positive real number, and p_1, p_2, q are continuous functions on $\overline{\Omega}$.

We study problem (3.14) under the following assumptions:

$$1 < p_2(x) < \min_{y\in\overline{\Omega}} q(y) \leq \max_{y\in\overline{\Omega}} q(y) < p_1(x), \quad \forall\, x \in \overline{\Omega} \tag{3.15}$$

and

$$\max_{y\in\overline{\Omega}} q(y) < p_2^\star(x), \quad \forall\, x \in \overline{\Omega}\,, \tag{3.16}$$

where $p_2^\star(x) := \frac{Np_2(x)}{N-p_2(x)}$ if $p_2(x) < N$ and $p_2^\star(x) = +\infty$ if $p_2(x) \geq N$.

Since $p_2(x) < p_1(x)$ for any $x \in \overline{\Omega}$ it follows that $W_0^{1,p_1(\cdot)}(\Omega)$ is continuously embedded in $W_0^{1,p_2(\cdot)}(\Omega)$. Thus, a solution for a problem of type (3.14) will be sought in the variable exponent space $W_0^{1,p_1(\cdot)}(\Omega)$.

We say that $\lambda \in \mathbf{R}$ is an *eigenvalue* of problem (3.14) if there exists $u \in W_0^{1,p_1(\cdot)}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u \nabla v\; dx - \lambda \int_\Omega |u|^{q(x)-2}uv\; dx = 0\,,$$

for all $v \in W_0^{1,p_1(\cdot)}(\Omega)$. We point out that if λ is an eigenvalue of problem (3.14) then the corresponding eigenfunction $u \in W_0^{1,p_1(\cdot)}(\Omega) \setminus \{0\}$ is a *weak solution* of problem (3.14).

Define

$$\lambda_1 := \inf_{u \in W_0^{1,p_1(\cdot)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}\; dx + \int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\; dx}{\displaystyle\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\; dx}\,.$$

Our main result is given by the following theorem.

Theorem 3.3. *Assume that conditions (3.15) and (3.16) are fulfilled. Then $\lambda_1 > 0$. Moreover, any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue of problem (3.14). Furthermore, there exists a positive constant λ_0 such that $\lambda_0 \leq \lambda_1$ and any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (3.14).*

Proof. Let E denote the generalized Sobolev space $W_0^{1,p_1(\cdot)}(\Omega)$. We denote by $\|\cdot\|_0$ the norm on $W_0^{1,p_1(\cdot)}(\Omega)$ and by $\|\cdot\|_1$ the norm on $W_0^{1,p_2(\cdot)}(\Omega)$.

Define functionals J, I, J_1, $I_1 : E \to \mathbf{R}$ by

$$J(u) = \int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}\; dx + \int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\; dx,$$

$$I(u) = \int_\Omega \frac{1}{q(x)} |u|^{q(x)}\, dx,$$

$$J_1(u) = \int_\Omega |\nabla u|^{p_1(x)}\, dx + \int_\Omega |\nabla u|^{p_2(x)}\, dx,$$

$$I_1(u) = \int_\Omega |u|^{q(x)}\, dx.$$

Standard arguments imply that $J,\ I \in C^1(E, \mathbf{R})$ and for all $u,\ v \in E$,

$$\langle J^{'}(u), v\rangle = \int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u \nabla v\, dx,$$

$$\langle I^{'}(u), v\rangle = \int_\Omega |u|^{q(x)-2} uv\, dx.$$

We split the proof of Theorem 3.3 into four steps.

• Step 1. We show that $\lambda_1 > 0$.

Since for any $x \in \overline{\Omega}$ we have $p_1(x) > q^+ \geq q(x) \geq q^- > p_2(x)$ we deduce that for any $u \in E$,

$$2(|\nabla u(x)|^{p_1(x)} + |\nabla u(x)|^{p_2(x)}) \geq |\nabla u(x)|^{q^+} + |\nabla u(x)|^{q^-}$$

and

$$|u(x)|^{q^+} + |u(x)|^{q^-} \geq |u(x)|^{q(x)}.$$

Integrating the above inequalities we find

$$2\int_\Omega (|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)})\, dx \geq \int_\Omega (|\nabla u|^{q^+} + |\nabla u|^{q^-})\, dx, \qquad \forall\, u \in E \tag{3.17}$$

and

$$\int_\Omega (|u|^{q^+} + |u|^{q^-})\, dx \geq \int_\Omega |u|^{q(x)}\, dx, \qquad \forall\, u \in E. \tag{3.18}$$

By Sobolev embeddings, there exist positive constants λ_{q^+} and λ_{q^-} such that

$$\int_\Omega |\nabla u|^{q^+}\, dx \geq \lambda_{q^+} \int_\Omega |u|^{q^+}\, dx, \qquad \forall\, u \in W_0^{1,q^+}(\Omega) \tag{3.19}$$

and

$$\int_\Omega |\nabla u|^{q^-}\, dx \geq \lambda_{q^-} \int_\Omega |u|^{q^-}\, dx, \qquad \forall\, u \in W_0^{1,q^-}(\Omega). \tag{3.20}$$

Using again the fact that $q^- \leq q^+ < p_1(x)$ for any $x \in \overline{\Omega}$ we deduce that E is continuously embedded in $W_0^{1,q^+}(\Omega)$ and in $W_0^{1,q^-}(\Omega)$. Thus, inequalities (3.19) and (3.20) hold true for any $u \in E$.

Using inequalities (3.19), (3.20) and (3.18) it is clear that there exists a positive constant μ such that

$$\int_\Omega (|\nabla u|^{q^+} + |\nabla u|^{q^-})\, dx \geq \mu \int_\Omega |u|^{q(x)}\, dx, \qquad \forall\, u \in E. \tag{3.21}$$

Next, inequalities (3.21) and (3.17) yield

$$\int_\Omega (|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)})\, dx \geq \frac{\mu}{2} \int_\Omega |u|^{q(x)}\, dx, \qquad \forall\, u \in E. \tag{3.22}$$

By relation (3.22) we deduce that

$$\lambda_0 = \inf_{v \in E \setminus \{0\}} \frac{J_1(v)}{I_1(v)} > 0 \tag{3.23}$$

and, thus,

$$J_1(u) \geq \lambda_0 I_1(u), \quad \forall\, u \in E. \tag{3.24}$$

The above inequality yields

$$p_1^+ \cdot J(u) \geq J_1(u) \geq \lambda_0 I_1(u) \geq \lambda_0 I(u) \quad \forall\, u \in E. \tag{3.25}$$

The last inequality assures that $\lambda_1 > 0$ and, thus, step 1 is verified.

• Step 2. We show that λ_1 is an eigenvalue of problem (3.14).

Lemma 3.2. *The following relations hold true:*

$$\lim_{\|u\|_0 \to \infty} \frac{J(u)}{I(u)} = \infty \tag{3.26}$$

and

$$\lim_{\|u\|_0 \to 0} \frac{J(u)}{I(u)} = \infty. \tag{3.27}$$

Proof. Since E is continuously embedded in $L^{q^\pm}(\Omega)$ it follows that there exist two positive constants c_1 and c_2 such that

$$\|u\|_0 \geq c_1 \cdot |u|_{q^+}, \quad \forall\, u \in E \tag{3.28}$$

and

$$\|u\|_0 \geq c_2 \cdot |u|_{q^-}, \quad \forall\, u \in E. \tag{3.29}$$

For any $u \in E$ with $\|u\|_0 > 1$ by relations (1.3), (3.18), (3.28), (3.29) we infer

$$\frac{J(u)}{I(u)} \geq \frac{\dfrac{\|u\|_0^{p_1^-}}{p_1^+}}{\dfrac{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}}{q^-}} \geq \frac{\dfrac{\|u\|_0^{p_1^-}}{p_1^+}}{\dfrac{c_1^{-q^+}\|u\|_0^{q^+} + c_2^{-q^-}\|u\|_0^{q^-}}{q^-}}.$$

Since $p_1^- > q^+ \geq q^-$, passing to the limit as $\|u\|_0 \to \infty$ in the above inequality we deduce that relation (3.26) holds true.

Next, let us remark that since $p_1(x) > p_2(x)$ for any $x \in \overline{\Omega}$, the space $W_0^{1,p_1(\cdot)}(\Omega)$ is continuously embedded in $W_0^{1,p_2(\cdot)}(\Omega)$. Thus, if $\|u\|_0 \to 0$ then $\|u\|_1 \to 0$.

The above remarks enable us to affirm that for any $u \in E$ with $\|u\|_0 < 1$ small enough we have $\|u\|_1 < 1$.

On the other hand, since (3.16) holds true we deduce that $W_0^{1,p_2(\cdot)}(\Omega)$ is continuously embedded in $L^{q^\pm}(\Omega)$. It follows that there exist two positive constants d_1 and d_2 such that

$$\|u\|_1 \geq d_1 \cdot |u|_{q^+}, \quad \forall\, u \in W_0^{1,p_2(\cdot)}(\Omega) \tag{3.30}$$

and

$$\|u\|_1 \geq d_2 \cdot |u|_{q^-}, \quad \forall\, u \in W_0^{1,p_2(\cdot)}(\Omega). \tag{3.31}$$

Thus, for any $u \in E$ with $\|u\|_0 < 1$ small enough, relations (1.4), (3.18), (3.30), (3.31) imply

$$\frac{J(u)}{I(u)} \geq \frac{\dfrac{\int_\Omega |\nabla u|^{p_2(x)}\,dx}{p_2^+}}{\dfrac{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}}{q^-}} \geq \frac{\dfrac{\|u\|_1^{p_2^+}}{p_2^+}}{\dfrac{d_1^{-q^+}\|u\|_1^{q^+} + d_2^{-q^-}\|u\|_1^{q^-}}{q^-}}.$$

Since $p_2^+ < q^- \leq q^+$, passing to the limit as $\|u\|_0 \to 0$ (and thus, $\|u\|_1 \to 0$) in the above inequality we deduce that relation (3.27) holds true. The proof of Lemma 3.2 is complete.

Lemma 3.3. *There exists $u \in E \setminus \{0\}$ such that $\frac{J(u)}{I(u)} = \lambda_1$.*

Proof. Let $\{u_n\} \subset E \setminus \{0\}$ be a minimizing sequence for λ_1, that is,

$$\lim_{n\to\infty} \frac{J(u_n)}{I(u_n)} = \lambda_1 > 0. \tag{3.32}$$

By relation (3.26) it is clear that $\{u_n\}$ is bounded in E. Since E is reflexive it follows that there exists $u \in E$ such that, up to a subsequence, $\{u_n\}$ converges weakly to u in E. On the other hand, standard arguments show that functional J is weakly lower semi-continuous. Thus, we find

$$\liminf_{n\to\infty} J(u_n) \geq J(u). \tag{3.33}$$

By the compact embedding theorem for spaces with variable exponent and assumption $1 \leq \max_{y\in\overline{\Omega}} q(y) < p_1(x)$ for all $x \in \overline{\Omega}$ (see (3.15)) it follows that E is compactly embedded in $L^{q(\cdot)}(\Omega)$. Thus, $\{u_n\}$ converges strongly in $L^{q(\cdot)}(\Omega)$. Then, by relation (1.5) it follows that

$$\lim_{n\to\infty} I(u_n) = I(u). \tag{3.34}$$

Relations (3.33) and (3.34) imply that if $u \not\equiv 0$ then

$$\frac{J(u)}{I(u)} = \lambda_1.$$

Thus, in order to conclude that the lemma holds true it is enough to show that u is not trivial. Assume by contradiction the contrary. Then u_n converges weakly to 0 in E and strongly in $L^{q(\cdot)}(\Omega)$. In other words, we will have

$$\lim_{n\to\infty} I(u_n) = 0. \tag{3.35}$$

Letting $\epsilon \in (0, \lambda_1)$ be fixed by relation (3.32) we deduce that for n large enough we have

$$|J(u_n) - \lambda_1 I(u_n)| < \epsilon I(u_n),$$

or

$$(\lambda_1 - \epsilon) I(u_n) < J(u_n) < (\lambda_1 + \epsilon) I(u_n).$$

Passing to the limit in the above inequalities and taking into account that relation (3.35) holds true we find

$$\lim_{n\to\infty} J(u_n)=0.$$

That fact combined with relation (1.5) implies that actually u_n converges strongly to 0 in E, i.e. $\lim_{n\to\infty}\|u_n\|_0=0$. By this information and relation (3.27) we get

$$\lim_{n\to\infty}\frac{J(u_n)}{I(u_n)}=\infty,$$

and this is a contradiction. Thus, $u\not\equiv 0$. The proof of Lemma 3.3 is complete.

By Lemma 3.3 we conclude that there exists $u\in E\setminus\{0\}$ such that

$$\frac{J(u)}{I(u)}=\lambda_1=\inf_{w\in E\setminus\{0\}}\frac{J(w)}{I(w)}. \tag{3.36}$$

Then, for any $v\in E$ we have

$$\frac{d}{d\epsilon}\frac{J(u+\epsilon v)}{I(u+\epsilon v)}\,|_{\epsilon=0}=0\,.$$

A simple computation yields

$$\int_\Omega(|\nabla u|^{p_1(x)-2}+|\nabla u|^{p_2(x)-2})\nabla u\nabla v\;dx\cdot I(u)-J(u)\cdot\int_\Omega|u|^{q(x)-2}uv\;dx=0,\quad\forall\,v\in E. \tag{3.37}$$

Relation (3.37) combined with the fact that $J(u)=\lambda_1 I(u)$ and $I(u)\neq 0$ implies the fact that λ_1 is an eigenvalue of problem (3.14). Thus, step 2 is verified.

• Step 3. We show that any $\lambda\in(\lambda_1,\infty)$ is an eigenvalue of problem (3.14).

Let $\lambda\in(\lambda_1,\infty)$ be arbitrary but fixed. Define $T_\lambda: E\to\mathbf{R}$ by

$$T_\lambda(u)=J(u)-\lambda I(u).$$

Clearly, $T_\lambda\in C^1(E,\mathbf{R})$ with

$$\langle T_\lambda^{'}(u),v\rangle=\langle J^{'}(u),v\rangle-\lambda\langle I^{'}(u),v\rangle,\quad\forall\,u\in E.$$

Thus, λ is an eigenvalue of problem (3.14) if and only if there exists $u_\lambda\in E\setminus\{0\}$ a critical point of T_λ.

With similar arguments as in the proof of relation (3.26) we can show that T_λ is coercive, i.e. $\lim_{\|u\|_0\to\infty}T_\lambda(u)=\infty$. On the other hand, as we have already remarked, functional T_λ is weakly lower semi-continuous. These two facts enable us to apply Theorem 1.2 in [75] in order to prove that there exists $u_\lambda\in E$ a global minimum point of T_λ and, thus, a critical point of T_λ. In order to conclude that step 4 holds true it is enough to show that u_λ is not trivial. Indeed, since $\lambda_1=\inf_{u\in E\setminus\{0\}}\frac{J(u)}{I(u)}$ and $\lambda>\lambda_1$ it follows that there exists $v_\lambda\in E$ such that

$$J(v_\lambda)<\lambda I(v_\lambda),$$

or

$$T_\lambda(v_\lambda)<0.$$

Thus,

$$\inf_E T_\lambda < 0$$

and we conclude that u_λ is a nontrivial critical point of T_λ, or λ is an eigenvalue of problem (3.14). Thus, step 3 is verified.

• Step 4. Any $\lambda \in (0, \lambda_0)$, where λ_0 is given by (3.23), is not an eigenvalue of problem (3.14).

Indeed, assuming by contradiction that there exists $\lambda \in (0, \lambda_0)$ an eigenvalue of problem (3.14) it follows that there exists $u_\lambda \in E \setminus \{0\}$ such that

$$\langle J'(u_\lambda), v\rangle = \lambda \langle I'(u_\lambda), v\rangle, \quad \forall\, v \in E.$$

Thus, for $v = u_\lambda$ we find

$$\langle J'(u_\lambda), u_\lambda\rangle = \lambda \langle I'(u_\lambda), u_\lambda\rangle,$$

that is,

$$J_1(u_\lambda) = \lambda I_1(u_\lambda).$$

The fact that $u_\lambda \in E \setminus \{0\}$ assures that $I_1(u_\lambda) > 0$. Since $\lambda < \lambda_0$, the above information yields

$$J_1(u_\lambda) \geq \lambda_0 I_1(u_\lambda) > \lambda I_1(u_\lambda) = J_1(u_\lambda).$$

Clearly, the above inequalities lead to a contradiction. Thus, step 4 is verified.

By steps 2, 3 and 4 we deduce that $\lambda_0 \leq \lambda_1$. The proof of Theorem 3.3 is now complete.

Remark 3.1. *At this stage we are not able to deduce whether $\lambda_0 = \lambda_1$ or $\lambda_0 < \lambda_1$. In the latter case an interesting question concerns the existence of eigenvalues of problem (3.14) in the interval $[\lambda_0, \lambda_1)$.*

3.5 An optimization result

In this section we are concerned with the study of the eigenvalue problem

$$\begin{cases} -\Delta_{p_1(x)} u - \Delta_{p_2(x)} u + V(x)|u|^{m(x)-2}u = \lambda(|u|^{q_1(x)-2} + |u|^{q_2(x)-2})u, & \text{for} \quad x \in \Omega \\ u = 0, & \text{for} \quad x \in \partial\Omega\,, \end{cases} \tag{3.38}$$

where $\Omega \subset \mathbf{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, λ is a real number, V is an indefinite sign-changing weight, and p_1, p_2, q_1, q_2, m are continuous functions on $\overline{\Omega}$. Problem (3.38) can be placed in the context of the eigenvalue problem studied in the above section since in the particular case when $q_1(x) = q_2(x) = q(x)$ for any $x \in \overline{\Omega}$ and $V \equiv 0$ in Ω it becomes problem (3.14). The form of problem (3.38) becomes a natural extension of problem (3.14) with the presence of the potential V in the left-hand side of the equation and by considering that in the right-hand side we have $q_1 \neq q_2$ on $\overline{\Omega}$.

More exactly, we study problem (3.38) when p_1, p_2, q_1, q_2, $m : \overline{\Omega} \to (1, \infty)$ are continuous functions satisfying the following hypotheses:

$$\max_{\overline{\Omega}} p_2 < \min_{\overline{\Omega}} q_2 \leq \max_{\overline{\Omega}} q_2 \leq \min_{\overline{\Omega}} m \leq \max_{\overline{\Omega}} m \leq \min_{\overline{\Omega}} q_1 \leq \max_{\overline{\Omega}} q_1 < \min_{\overline{\Omega}} p_1\,, \tag{3.39}$$

$$\max_{\overline{\Omega}} q_1 < p_2^\star(x) := \begin{cases} \dfrac{Np_2(x)}{N-p_2(x)} & \text{if } p_2(x) < N \\ +\infty & \text{if } p_2(x) \geq N. \end{cases} \tag{3.40}$$

We assume that the potential $V : \Omega \to \mathbf{R}$ satisfies

$$V \in L^{r(\cdot)}(\Omega), \quad \text{with } r \in C(\overline{\Omega}) \text{ and } r(x) > \frac{N}{\min_{\overline{\Omega}} m} \quad \forall\, x \in \overline{\Omega}\,. \tag{3.41}$$

Condition (3.39) which describes the competition between the growth rates involved in equation (3.38) represents the *key* of the present study since it establishes a balance between all the variable exponents involved in the problem. Such a balance is essential since our setting assumes a non-homogeneous eigenvalue problem for which a minimization technique based on the Lagrange multiplier theorem can not be applied in order to find (principal) eigenvalues (unlike the case offered by the homogeneous operators). Thus, in the case of nonlinear non-homogeneous eigenvalue problems the classical theory used in the homogeneous case does not work entirely, but some of its ideas can still be useful and some particular results can still be obtained in some aspects while in other aspects entirely new phenomena can occur. To focus on our case, condition (3.39) together with conditions (3.40) and (3.41) imply

$$\lim_{\|u\|_{p_1(\cdot)}\to 0} \frac{\displaystyle\int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}\,dx + \int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\,dx + \int_\Omega \frac{V(x)}{m(x)}|u|^{m(x)}\,dx}{\displaystyle\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}\,dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}\,dx} = \infty$$

and

$$\lim_{\|u\|_{p_1(\cdot)}\to \infty} \frac{\displaystyle\int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}\,dx + \int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\,dx + \int_\Omega \frac{V(x)}{m(x)}|u|^{m(x)}\,dx}{\displaystyle\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}\,dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}\,dx} = \infty\,,$$

where $\|\cdot\|_{p_1(\cdot)}$ stands for the norm in the variable exponent Sobolev space $W_0^{1,p_1(\cdot)}(\Omega)$. In other words, the absence of homogeneity is balanced by the behavior (actually, the blow-up) of the Rayleigh quotient associated to problem (3.38) in the origin and at infinity. The consequences of the above remarks is that the infimum of the Rayleigh quotient associated to problem (3.38) is a real number, i.e.

$$\inf_{u\in W_0^{1,p_1(\cdot)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}\,dx + \int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\,dx + \int_\Omega \frac{V(x)}{m(x)}|u|^{m(x)}\,dx}{\displaystyle\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}\,dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}\,dx} \in \mathbf{R}\,, \tag{3.42}$$

and it will be attained for a function $u_0 \in W_0^{1,p_1(\cdot)}(\Omega) \setminus \{0\}$. Moreover, the value in (3.42) represents an eigenvalue of problem (3.38) with the corresponding eigenfunction u_0. However, at this stage we can not say if the eigenvalue described above is the lowest eigenvalue of problem (3.38) or not, even if we are able to show that any λ small enough is not an eigenvalue of (3.38). At the moment this remains an open question. On the other hand, we can prove that any λ larger than the value given by relation (3.42) is also an eigenvalue of problem (3.38). Thus, we conclude that problem (3.38) possesses a continuous family of eigenvalues.

Related with the above ideas we will also discuss the *optimization* of the eigenvalues described by relation (3.42) with respect to the potential V, providing that V belongs to a bounded, closed and convex subset of $L^{r(\cdot)}(\Omega)$ (where $r(x)$ is given by relation (3.41)). By optimization we understand the existence of some potentials $V_\star$ and $V^\star$ such that the eigenvalue described in relation (3.42) is minimal or maximal with respect to the set where V lies. The results that we will obtain in the context of optimization of eigenvalues are motivated by the above advances in this field in the case of homogeneous (linear or nonlinear) eigenvalue problems. We refer mainly to the studies in M. S. Asbaugh & E. M. Harrell [9], H. Egnell [24] and J. F. Bonder & L. M. Del Pezzo [12] where different optimization problems of the principal eigenvalue of some homogeneous operators were studied.

Since $p_2(x) < p_1(x)$ for any $x \in \overline{\Omega}$ it follows that $W_0^{1,p_1(\cdot)}(\Omega)$ is continuously embedded in $W_0^{1,p_2(\cdot)}(\Omega)$. Thus, a solution for a problem of type (3.38) will be sought in the variable exponent space $W_0^{1,p_1(\cdot)}(\Omega)$.

We say that $\lambda \in \mathbf{R}$ is an *eigenvalue* of problem (3.38) if there exists $u \in W_0^{1,p_1(\cdot)}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega (|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u \nabla v\, dx + \int_\Omega V(x)|u|^{m(x)-2}uv\, dx - \lambda \int_\Omega (|u|^{q_1(x)-2} + |u|^{q_2(x)-2})uv\, dx = 0\,,$$

for all $v \in W_0^{1,p_1(\cdot)}(\Omega)$. We point out that if λ is an eigenvalue of problem (3.38) then the corresponding *eigenfunction* $u \in W_0^{1,p_1(\cdot)}(\Omega) \setminus \{0\}$ is a *weak solution* of problem (3.38).

For each potential $V \in L^{r(\cdot)}(\Omega)$ we define

$$E(V) := \inf_{u \in W_0^{1,p_1(\cdot)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}\, dx + \int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\, dx + \int_\Omega \frac{V(x)}{m(x)}|u|^{m(x)}\, dx}{\displaystyle\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}\, dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}\, dx}$$

and

$$F(V) := \inf_{u \in W_0^{1,p_1(\cdot)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^{p_1(x)}\, dx + \int_\Omega |\nabla u|^{p_2(x)}\, dx + \int_\Omega V(x)|u|^{m(x)}\, dx}{\displaystyle\int_\Omega |u|^{q_1(x)}\, dx + \int_\Omega |u|^{q_2(x)}\, dx}\,.$$

Thus, we can define a function $E : L^{r(\cdot)}(\Omega) \to \mathbf{R}$.

The first result of this section is given by the following theorem.

Theorem 3.4. *Assume that conditions (3.39), (3.40) and (3.41) are fulfilled. Then $E(V)$ is an eigenvalue of problem (3.38). Moreover, there exists $u \in W_0^{1,p_1(x)}(\Omega) \setminus \{0\}$ an eigenfunction corresponding to eigenvalue $E(V)$ such that*

$$E(V) = \frac{\displaystyle\int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}\, dx + \int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\, dx + \int_\Omega \frac{V(x)}{m(x)}|u|^{m(x)}\, dx}{\displaystyle\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}\, dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}\, dx}\,.$$

Furthermore, $F(V) \le E(V)$, each $\lambda \in (E(V), \infty)$ is an eigenvalue of problem (3.38), while each $\lambda \in (-\infty, F(V))$ is not an eigenvalue of problem (3.38).

Next, we point out the case of convex, bounded and closed subsets of $L^{r(\cdot)}(\Omega)$ where function E defined above is bounded from below and attains its minimum. The result is the following:

Theorem 3.5. *Assume that conditions (3.39), (3.40) and (3.41) are fulfilled. Assume that S is a convex, bounded and closed subset of $L^{r(x)}(\Omega)$. Then there exists $V_\star \in S$ which minimizes $E(V)$ on S, i.e.*

$$E(V_\star) = \inf_{V \in S} E(V)\,.$$

Finally, we will focus our attention on the particular case when set S from Theorem 3.5 is a ball in $L^{r(\cdot)}(\Omega)$. Thus, we will denote each closed ball centered in the origin of radius R from $L^{r(\cdot)}(\Omega)$ by $\overline{B}_R(0)$, i.e.

$$\overline{B}_R(0) := \{u \in L^{r(\cdot)}(\Omega);\;\; |u|_{r(x)} \le R\}\,.$$

By Theorem 3.5 we can define function $E_\star : [0,\infty) \to \mathbf{R}$ by

$$E_\star(R) = \min_{V \in \overline{B}_R(0)} E(V)\,.$$

Our result on function $E_\star$ is given by the following theorem:

Theorem 3.6. *a) Function $E_\star$ is not constant and decreases monotonically.*
b) Function $E_\star$ is continuous.

On the other hand, we point out that similar results as those of Theorems 3.5 and 3.6 can be obtained if we notice that on each convex, bounded and closed subset of $L^{r(\cdot)}(\Omega)$ function E defined in Theorem 3.4 is also bounded from above and attains its maximum. It is also easy to remark that we can define a function $E^\star : [0,\infty) \to \mathbf{R}$ by

$$E^\star(R) = \max_{V \in \overline{B}_R(0)} E(V)\,,$$

which has similar properties as $E_\star$.

3.6 The case of unbounded domains

Consider the eigenvalue problem

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u + |u|^{q(x)-2}u = \lambda g(x)|u|^{r(x)-2}u & \text{for} \quad x \in \Omega \\ u = 0 \qquad \text{for} \quad x \in \partial\Omega\,, \end{cases} \tag{3.43}$$

where Ω is a smooth exterior domain in $\mathbf{R}^N$ ($N \ge 3$), that is, Ω is the complement of a bounded domain with Lipschitz boundary. Mappings p, q, $r : \overline{\Omega} \to [2,\infty)$ are Lipschitz continuous functions while $g : \overline{\Omega} \to [0,\infty)$ is a measurable function for which there exists a nonempty set $\Omega_0 \subset \Omega$ such that $g(x) > 0$ for any $x \in \Omega_0$, and $\lambda \ge 0$ is a real number.

Assume that functions p, q and r satisfy hypotheses

$$2 \le p^- \le p^+ < N\,, \tag{3.44}$$

$$p^+ < r^- \le r^+ < q^- \le q^+ < \frac{Np^-}{N-p^-}\,. \tag{3.45}$$

Furthermore, we assume that function $g(x)$ satisfies the hypothesis

$$g \in L^\infty(\Omega) \cap L^{p_0(\cdot)}(\Omega)\,, \tag{3.46}$$

where $p_0(x) = p^\star(x)/(p^\star(x) - r^-)$ for any $x \in \overline{\Omega}$.

Obviously, the natural space where we should seek solutions for problem (3.43) is space $W_0^{1,p(\cdot)}(\Omega)$.

We say that $\lambda \in \mathbf{R}$ is an eigenvalue of problem (3.43) if there exists $u \in W_0^{1,p(\cdot)}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv + |u|^{q(x)-2}uv)\,dx - \lambda \int_\Omega g(x)|u|^{r(x)-2}uv\,dx = 0,$$

for all $v \in W_0^{1,p(\cdot)}(\Omega)$. We point out that if λ is an eigenvalue of problem (3.43) then the corresponding $u \in W_0^{1,p(\cdot)}(\Omega) \setminus \{0\}$ is a weak solution of (3.43).

Define

$$\lambda_1 := \inf_{u \in W_0^{1,p(\cdot)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx}{\displaystyle\int_\Omega \frac{g(x)}{r(x)}|u|^{r(x)}\,dx}$$

and

$$\lambda_0 := \inf_{u \in W_0^{1,p(\cdot)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx + \int_\Omega |u|^{q(x)}\,dx}{\displaystyle\int_\Omega g(x)|u|^{r(x)}\,dx}\,.$$

Our main result on problem (3.43) is given by the following theorem.

Theorem 3.7. *Let Ω be an exterior domain with Lipschitz boundary in $\mathbf{R}^N$, where $N \ge 3$. Suppose that p, q, $r : \overline{\Omega} \to [2,\infty)$ are Lipschitz continuous functions and $g : \overline{\Omega} \to [0,\infty)$ is a measurable function for which there exists a nonempty set $\Omega_0 \subset \Omega$ such that $g > 0$ in Ω_0. Assume conditions (3.44), (3.45), and (3.46) are fulfilled.*

Then

$$0 < \lambda_0 \le \lambda_1\,.$$

Furthermore, each $\lambda \in [\lambda_1, \infty)$ is an eigenvalue of problem (3.43) while any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (3.43).

At this stage we are not able to deduce whether $\lambda_0 = \lambda_1$ or $\lambda_0 < \lambda_1$. In the latter case an interesting open problem concerns the existence of eigenvalues of problem (3.43) in the interval $[\lambda_0, \lambda_1)$.

3.7 The anisotropic case

The purpose of this section is to analyze the nonhomogeneous anisotropic eigenvalue problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i}u|^{p_i(x)-2}\,\partial_{x_i}u\right) = \lambda|u|^{q(x)-2}u & \text{in} \quad \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{3.47}$$

where $\Omega\subset \mathbf{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary, λ is a positive number, and p_i, q are continuous functions on $\overline{\Omega}$ such that $2\leq p_i(x)<N$ and $q(x)>1$ for any $x\in\overline{\Omega}$ and $i\in\{1,...,N\}$.

The natural function space where problem (3.47) should be analyzed is the anisotropic variable exponent Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$. For definitions, notations and properties of anisotropic variable exponent spaces we refer to Chapter 1.

We say that $\lambda\in\mathbf{R}$ is an *eigenvalue* of problem (3.47) if there exists $u\in W_0^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}$ such that

$$\int_\Omega \left\{\sum_{i=1}^{N}|\partial_{x_i}u|^{p_i(x)-2}\,\partial_{x_i}u\partial_{x_i}\varphi-\lambda|u|^{q(x)-2}u\varphi\right\}dx=0$$

for all $\varphi\in W_0^{1,\vec{p}(\cdot)}(\Omega)$. For $\lambda\in\mathbf{R}$ an eigenvalue of problem (3.47) function u from the above definition will be called a *weak solution* of problem (3.47) corresponding to eigenvalue λ.

The main results on problem (3.47) are listed below:

Theorem 3.8. *Assume that function $q\in C(\overline{\Omega})$ verifies hypothesis*

$$P_+^+<\min_{x\in\overline{\Omega}}q(x)\leq\max_{x\in\overline{\Omega}}q(x)<P_-^\star. \tag{3.48}$$

Then for any $\lambda>0$ problem (3.47) possesses a nontrivial weak solution.

Theorem 3.9. *If $q\in C(\overline{\Omega})$ satisfies inequalities*

$$1<\min_{x\in\overline{\Omega}}q(x)\leq\max_{x\in\overline{\Omega}}q(x)<P_-^-\,, \tag{3.49}$$

then for any $\lambda>0$ problem (3.47) possesses a nontrivial weak solution.

Theorem 3.10. *If $q\in C(\overline{\Omega})$, with*

$$1<\min_{x\in\overline{\Omega}}q(x)<P_-^- \quad \textit{and} \quad \max_{x\in\overline{\Omega}}q(x)<P_{-,\infty}, \tag{3.50}$$

then there exists $\lambda^\star>0$ such that for any $\lambda\in(0,\lambda^\star)$ problem (3.47) possesses a nontrivial weak solution.

Remark 3.2. *If $q\in C(\overline{\Omega})$ verifies (3.49) then it satisfies (3.50). On the other hand, we point out that the result of Theorem 3.10 holds true in situations that extend relation (3.49) since in relation (3.50) we could have*

$$1<\min_{x\in\overline{\Omega}}q(x)<P_-^-<\max_{x\in\overline{\Omega}}q(x)<P_{-,\infty}.$$

In order to enunciate the next result on problem (3.47) we consider the following assumptions on functions $p_i,\ q$:

(a1) Assume that there exists $j \in \{1,...,N\}$ such that $q(x) = q(x_1,...,x_{j-1},x_{j+1},...,x_N)$ (i.e. q is independent of x_j) and $p_j(x) = q(x)$ for all $x \in \overline{\Omega}$.

(a2) Assume that there exists $k \in \{1,...,N\}$ ($k \neq j$ with j given in (a1)) such that

$$\max_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p_k(x)\,.$$

Define the Rayleigh type quotients λ_0 and λ_1 associated with problem (3.47) by

$$\lambda_0 = \inf_{u\in W_0^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \sum_{i=1}^N |\partial_i u|^{p_i(x)}\,dx}{\displaystyle\int_\Omega |u|^{q(x)}\,dx}, \quad \lambda_1 = \inf_{u\in W_0^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)}|\partial_i u|^{p_i(x)}\,dx}{\displaystyle\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx}\,.$$

Theorem 3.11. *Assume conditions (a1) and (a2) are fulfilled. Then* $0 < \lambda_0 \leq \lambda_1$ *and every* $\lambda \in (\lambda_1,\infty)$ *is an eigenvalue of problem (3.47), while no* $\lambda \in (0,\lambda_0)$ *can be an eigenvalue of problem (3.47).*

Remark 3.3. *At this stage we are not able to say whether* $\lambda_0 = \lambda_1$ *or* $\lambda_0 < \lambda_1$*. In the latter case an interesting question concerns the existence of eigenvalues of problem (3.47) in the interval* $[\lambda_0,\lambda_1]$.

We note that we can also obtain results of the type of those enunciated above by replacing operator $\displaystyle\sum_{i=1}^N \partial_{x_i}\left(|\partial_{x_i}u|^{p_i(x)-2}\,\partial_{x_i}u\right)$ in problem (3.47) with a more general one, namely an operator of the type $\displaystyle\sum_{i=1}^N \partial_{x_i}\left(a_i(x,\partial_{x_i}u)\right)$, where for each $i \in \{1,...,N\}$ we assume that $a_i(x,t) : \overline{\Omega}\times\mathbf{R} \to \mathbf{R}$ is a continuous function. For each $i \in \{1,...,N\}$ define $A_i(x,t) : \overline{\Omega}\times\mathbf{R} \to \mathbf{R}$, a primitive with respect to variable t of $a_i(x,t)$, i.e. $A_i(x,t) = \int_0^t a_i(x,s)\,ds$. The above results can be extended if we suppose that a_i and A_i satisfy the following hypotheses:

(A1) There exists a positive constant $c_{1,i}$ and a continuous function $p_i(x) : \overline{\Omega} \to [2,\infty)$ such that

$$|a_i(x,t)| \leq c_{1,i}(1+|t|^{p_i(x)-1}),$$

for all $x \in \overline{\Omega}$ and $t \in \mathbf{R}$.

(A2) There exists $k_i > 0$ such that

$$A_i\left(x, \frac{t+s}{2}\right) \leq \frac{1}{2}A_i(x,t) + \frac{1}{2}A_i(x,s) - k_i|t-s|^{p_i(x)}$$

for all $x \in \overline{\Omega}$ and $t,\ s \in \mathbf{R}$, where $p_i(x)$ is given in (A2).

(A3) The following inequalities hold true

$$|t|^{p_i(x)} \leq a_i(x,t)\cdot t \leq p_i(x)\ A_i(x,t),$$

for all $x \in \overline{\Omega}$ and $t \in \mathbf{R}$, where $p_i(x)$ is given in (A1).

Examples.
1. Set $A_i(x,t) = \frac{1}{p_i(x)}|t|^{p_i(x)}$, $a_i(x,t) = |t|^{p_i(x)-2}t$, where $p_i(x) \geq 2$. Such a function contributes to equation (3.47) with the term

$$\partial_{x_i}(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u).$$

2. Set $A_i(x,t) = \frac{1}{p_i(x)}[(1+t^2)^{p_i(x)/2} - 1]$, $a_i(x,t) = (1+t^2)^{(p_i(x)-2)/2}t$, where $p_i(x) \geq 2$. Such a function contributes to equation (3.47) with the term

$$\partial_{x_i}((1+|\partial_{x_i}u|^2)^{(p_i(x)-2)/2}\partial_{x_i}u).$$

Chapter 4

Dirichlet eigenvalue problems in Orlicz-Sobolev spaces

4.1 Eigenvalue problem $-\text{div}(a(|\nabla u|)\nabla u) = \lambda|u|^{q(x)-2}u$

4.1.1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^N$ ($N \geq 3$) with smooth boundary $\partial\Omega$. In this section we are concerned with the following eigenvalue problem:

$$\begin{cases} -\text{div}(a(|\nabla u|)\nabla u) = \lambda|u|^{q(x)-2}u, & \text{for} \quad x \in \Omega \\ u = 0, & \text{for} \quad x \in \partial\Omega \,. \end{cases} \tag{4.1}$$

We assume that function $a : (0, \infty) \to \mathbf{R}$ is such that mapping $\varphi : \mathbf{R} \to \mathbf{R}$ defined by

$$\varphi(t) = \begin{cases} a(|t|)t, & \text{for} \quad t \neq 0 \\ 0, & \text{for} \quad t = 0 \,, \end{cases}$$

is an odd, increasing homeomorphism from $\mathbf{R}$ onto $\mathbf{R}$. We also suppose throughout this section that $\lambda > 0$ and $q : \overline{\Omega} \to (1, \infty)$ is a continuous function.

Since the operator in the divergence form is nonhomogeneous we introduce an Orlicz-Sobolev space setting for problems of this type (see Chapter 1 for definitions, notations and properties of Orlicz-Sobolev spaces). Thus, the space where we analyze problem (4.1) is space $W_0^1 L_\Phi(\Omega)$, where

$$\Phi(t) = \int_0^t \varphi(s)\, ds, \qquad \text{for all } t \in \mathbf{R}\,.$$

Here, we assume that condition (1.10) is fulfilled and

$$\lim_{t\to 0} \int_t^1 \frac{\Phi^{-1}(s)}{s^{(N+1)/N}}\, ds < \infty, \quad \text{and} \quad \lim_{t\to\infty} \int_1^t \frac{\Phi^{-1}(s)}{s^{(N+1)/N}}\, ds = \infty. \tag{4.2}$$

We say that $\lambda \in \mathbf{R}$ is an eigenvalue of problem (4.1) if there exists $u \in W_0^1 L_\Phi(\Omega) \setminus \{0\}$ such that

$$\int_\Omega a(|\nabla u|)\nabla u \nabla v \, dx - \lambda \int_\Omega |u|^{q(x)-2} uv \, dx = 0,$$

for all $v \in W_0^1 L_\Phi(\Omega)$. We point out that if λ is an eigenvalue of problem (4.1) then the corresponding $u \in W_0^1 L_\Phi(\Omega) \setminus \{0\}$ is a weak solution of (4.1), called an eigenvector of equation (4.1) corresponding to eigenvalue λ.

Our first main result shows that, under certain circumstances, any positive and sufficiently small λ is an eigenvalue of (4.1).

Theorem 4.1. *Assume that relation (4.2) is fulfilled and furthermore*

$$1 < \inf_{x \in \Omega} q(x) < (p)_0 \, , \tag{4.3}$$

and

$$\lim_{t \to \infty} \frac{|t|^{q^+}}{\Phi_\star(kt)} = 0, \quad \text{for all } k > 0 \, , \tag{4.4}$$

where $\Phi_\star$ stands for the Orlicz-Sobolev conjugate of Φ, that is

$$\Phi_\star^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{(N+1)/N}} \, ds \, .$$

Then there exists $\lambda^\star > 0$ such that any $\lambda \in (0, \lambda^\star)$ is an eigenvalue of problem (4.1).

The above result implies

$$\inf_{u \in W_0^1 L_\Phi(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega \Phi(|\nabla u|) \, dx}{\displaystyle\int_\Omega |u|^{q(x)} \, dx} = 0.$$

The second main result of this section asserts that in certain cases the set of eigenvalues may coincide with the *whole* positive semiaxis.

Theorem 4.2. *Assume that relations (4.2) and (4.4) are fulfilled and furthermore*

$$\sup_{x \in \Omega} q(x) < (p)_0. \tag{4.5}$$

Then every $\lambda > 0$ is an eigenvalue of problem (4.1). Moreover, for any $\lambda > 0$ there exists a sequence of eigenvectors $\{u_n\} \subset E$ such that $\lim_{n \to \infty} u_n = 0$ in $W_0^1 L_\Phi(\Omega)$.

Remark 1. Relations (4.2) and (4.4) enable us to apply Theorem 2.2 in [33] (see also Theorem 8.33 in [3]) in order to obtain that $W_0^1 L_\Phi(\Omega)$ is compactly embedded in $L^{q^+}(\Omega)$. This fact combined with the continuous embedding of $L^{q^+}(\Omega)$ in $L^{q(\cdot)}(\Omega)$ ensures that $W_0^1 L_\Phi(\Omega)$ is compactly embedded in $L^{q(\cdot)}(\Omega)$.

Remark 2. The conclusion of Theorems 4.1 and 4.2 still remains valid if we replace hypothesis (4.4) in Theorems 4.1 and 4.2 by the following relation

$$N < (p)_0 < \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}. \tag{4.6}$$

Indeed, using Lemma D.2 in [19], it follows that $W_0^1 L_\Phi(\Omega)$ is continuously embedded in $W_0^{1,(p)_0}(\Omega)$. On the other hand, since we assume $(p)_0 > N$, we deduce that $W_0^{1,(p)_0}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Thus, we obtain that $W_0^1 L_\Phi(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Since Ω is bounded it follows that $W_0^1 L_\Phi(\Omega)$ is continuously embedded in $L^{q(\cdot)}(\Omega)$.

4.1.2 Proof of Theorem 4.1

Let E denote the Orlicz-Sobolev space $W_0^1 L_\Phi(\Omega)$.

For any $\lambda > 0$ the energy functional $J_\lambda : E \to \mathbf{R}$ corresponding to problem (4.1) is defined by

$$J_\lambda(u) = \int_\Omega \Phi(|\nabla u|)\, dx - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)}\, dx.$$

Standard arguments imply that $J_\lambda \in C^1(E, \mathbf{R})$ and

$$\langle J_\lambda^{'}(u), v\rangle = \int_\Omega a(|\nabla u|)\nabla u \nabla v\, dx - \lambda \int_\Omega |u|^{q(x)-2} uv\, dx,$$

for all $u,\ v \in E$. Thus the weak solutions of (4.1) coincide with the critical points of J_λ. If such a weak solution exists and is nontrivial then the corresponding λ is an eigenvalue of problem (4.1).

Lemma 4.1. *There is some $\lambda^\star > 0$ such that for any $\lambda \in (0, \lambda^\star)$ there exist $\rho,\ \alpha > 0$ such that $J_\lambda(u) \geq \alpha > 0$ for any $u \in E$ with $\|u\|_{0,\Phi} = \rho$.*

Proof. By the definition of $(p)^0$ and since $\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tau^{(p)_0}\Phi(t/\tau)\right) \geq 0$ we obtain

$$\Phi(t) \geq \tau^{(p)^0}\Phi(t/\tau), \quad \forall\, t > 0 \text{ and } \tau \in (0,1]\,,$$

(see page 44 in [18]). Combining this fact with Proposition 6 in [71, page 77] we find that

$$\int_\Omega \Phi(|\nabla u(x)|)\, dx \geq \|u\|_{0,\Phi}^{(p)^0}, \quad \forall\, u \in E \text{ with } \|u\|_{0,\Phi} < 1. \tag{4.7}$$

On the other hand, since E is continuously embedded in $L^{q(x)}(\Omega)$, there exists a positive constant c_1 such that

$$|u|_{q(\cdot)} \leq c_1 \|u\|_{0,\Phi}, \quad \forall\, u \in E. \tag{4.8}$$

We fix $\rho \in (0,1)$ such that $\rho < 1/c_1$. Then relation (4.8) implies

$$|u|_{q(\cdot)} < 1, \quad \forall\, u \in E, \text{ with } \|u\|_{0,\Phi} = \rho. \tag{4.9}$$

Furthermore, relation (1.4) yields

$$\int_\Omega |u|^{q(x)}\,dx \le |u|_{q(\cdot)}^{q^-}, \quad \forall\, u \in E, \text{ with } \|u\|_{0,\Phi} = \rho. \tag{4.10}$$

Relations (4.8) and (4.10) imply

$$\int_\Omega |u|^{q(x)}\,dx \le c_1^{q^-}\|u\|_{0,\Phi}^{q^-}, \quad \forall\, u \in E, \text{ with } \|u\|_{0,\Phi} = \rho. \tag{4.11}$$

Taking into account relations (4.7), (1.4) and (4.11) we deduce that for any $u \in E$ with $\|u\|_{0,\Phi} = \rho$ the following inequalities hold true

$$J_\lambda(u) \ge \|u\|_{0,\Phi}^{(p)^0} - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}\,dx = \rho^{q^-}\left(\rho^{(p)^0-q^-} - \frac{\lambda}{q^-}c_1^{q^-}\right).$$

We point out that by relation (4.3) and the definition of $(p)^0$ we have $q^- < l \le (p)^0$. By the above inequality we remark that if we define

$$\lambda^\star = \frac{\rho^{(p)^0-q^-}}{2}\cdot\frac{q^-}{c_1^{q^-}} \tag{4.12}$$

then for any $\lambda \in (0,\lambda^\star)$ and any $u \in E$ with $\|u\|_{0,\Phi} = \rho$ there exists $\alpha = \frac{\rho^{(p)^0}}{2} > 0$ such that

$$J_\lambda(u) \ge \alpha > 0.$$

The proof of Lemma 4.1 is complete.

Lemma 4.2. *There exists $\varphi \in E$ such that $\varphi \ge 0$, $\varphi \ne 0$ and $J_\lambda(t\varphi) < 0$, for $t > 0$ small enough.*

Proof. Assumption (4.3) implies that $q^- < (p)_0$. Let $\epsilon_0 > 0$ be such that $q^- + \epsilon_0 < (p)_0$. On the other hand, since $q \in C(\overline{\Omega})$ it follows that there exists an open set $\Omega_0 \subset \Omega$ such that $|q(x) - q^-| < \epsilon_0$ for all $x \in \Omega_0$. Thus, we conclude that $q(x) \le q^- + \epsilon_0 < (p)_0$ for all $x \in \Omega_0$.

Let $\psi \in C_0^\infty(\Omega)$ be such that $\mathrm{supp}(\psi) \supset \overline{\Omega}_0$, $\psi(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \le \psi \le 1$ in Ω.

We also point out that there exists $t_0 \in (0,1)$ such that for any $t \in (0,t_0)$ we have

$$\|t|\nabla\psi|\|_\Phi = t\|\psi\|_{0,\Phi} < 1.$$

Taking into account all the above information and using Lemma C.9 in [19] we have

$$\begin{aligned} J_\lambda(t\psi) &= \int_\Omega \Phi(t|\nabla\psi(x)|)\,dx - \lambda\int_\Omega \frac{t^{q(x)}}{q(x)}|\psi|^{q(x)}\,dx \\ &\le \int_\Omega \Phi(t|\nabla\psi(x)|)\,dx - \frac{\lambda}{q^+}\int_\Omega t^{q(x)}|\psi|^{q(x)}\,dx \\ &\le \int_\Omega \Phi(t|\nabla\psi(x)|)\,dx - \frac{\lambda}{q^+}\int_{\Omega_0} t^{q(x)}|\psi|^{q(x)}\,dx \\ &\le t^{(p)_0}\|\psi\|_{0,\Phi}^{(p)_0} - \frac{\lambda\cdot t^{q^-+\epsilon_0}}{q^+}|\Omega_0|, \end{aligned}$$

for any $t \in (0,1)$, where $|\Omega_0|$ denotes the Lebesgue measure of Ω_0. Therefore

$$J_\lambda(t\psi) < 0$$

for $t < \delta^{1/((p)_0 - q^- - \epsilon_0)}$, where

$$0 < \delta < \min\left\{ t_0, \frac{\frac{\lambda}{q^+}|\Omega_0|}{\|\psi\|_{0,\Phi}^{(p)_0}} \right\}.$$

The proof of Lemma 4.2 is complete.

Proof of Theorem 4.1. Let $\lambda^\star > 0$ be defined as in (4.12) and $\lambda \in (0, \lambda^\star)$. By Lemma 4.1 it follows that on the boundary of the ball centered at the origin and of radius ρ in E, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} J_\lambda > 0. \tag{4.13}$$

On the other hand, by Lemma 4.2, there exists $\varphi \in E$ such that $J_\lambda(t\varphi) < 0$ for all $t > 0$ small enough. Moreover, relations (4.7), (4.11) and (1.4) imply that for any $u \in B_\rho(0)$ we have

$$J_\lambda(u) \geq \|u\|_{0,\Phi}^{(p)^0} - \frac{\lambda}{q^-} c_1^{q^-} \|u\|_{0,\Phi}^{q^-}.$$

It follows that

$$-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} J_\lambda < 0.$$

We let now $0 < \epsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda$. Applying Ekeland's variational principle [25] to functional $J_\lambda : \overline{B_\rho(0)} \to \mathbf{R}$, we find $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$\begin{aligned} J_\lambda(u_\epsilon) &< \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon \\ J_\lambda(u_\epsilon) &< J_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|_{0,\Phi}, \quad u \neq u_\epsilon. \end{aligned}$$

Since

$$J_\lambda(u_\epsilon) \leq \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon \leq \inf_{B_\rho(0)} J_\lambda + \epsilon < \inf_{\partial B_\rho(0)} J_\lambda\,,$$

we deduce that $u_\epsilon \in B_\rho(0)$. Now, we define $I_\lambda : \overline{B_\rho(0)} \to \mathbf{R}$ by $I_\lambda(u) = J_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|_{0,\Phi}$. It is clear that u_ϵ is a minimum point of I_λ and, thus,

$$\frac{I_\lambda(u_\epsilon + t\cdot v) - I_\lambda(u_\epsilon)}{t} \geq 0$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{J_\lambda(u_\epsilon + t\cdot v) - J_\lambda(u_\epsilon)}{t} + \epsilon \cdot \|v\|_{0,\Phi} \geq 0.$$

Letting $t \to 0$ it follows that $\langle J_\lambda'(u_\epsilon), v\rangle + \epsilon \cdot \|v\|_{0,\Phi} > 0$ and we infer that $\|J_\lambda'(u_\epsilon)\| \leq \epsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$J_\lambda(w_n) \to \underline{c} \quad \text{and} \quad J_\lambda^{'}(w_n) \to 0. \tag{4.14}$$

It is clear that $\{w_n\}$ is bounded in E. Thus, there exists $w \in E$ such that, up to a subsequence, $\{w_n\}$ converges weakly to w in E. By Remark 2 we deduce that E is compactly embeddded in $L^{q(x)}(\Omega)$, hence $\{w_n\}$ converges strongly to w in $L^{q(x)}(\Omega)$. So, by relations (1.5) and Hölder's inequality for variable exponent spaces (see e.g. [43]),

$$\lim_{n\to\infty}\int_\Omega |w_n|^{q(x)}\,dx = \int_\Omega |w|^{q(x)}\,dx \quad \text{and} \quad \lim_{n\to\infty}\int_\Omega |w_n|^{q(x)-2}w_n v\,dx = \int_\Omega |w|^{q(x)-2}wv\,dx$$

for any $v \in E$.

We conclude that w is a nontrivial weak solution for problem (4.1) and, thus, any $\lambda \in (0, \lambda^\star)$ is an eigenvalue of problem (4.1). Similar arguments as those used on page 50 in [18] imply that $\{w_n\}$ converges strongly to w in E. So, by (4.14),

$$J_\lambda(w) = \underline{c} < 0 \quad \text{and} \quad J_\lambda^{'}(w) = 0. \tag{4.15}$$

The proof of Theorem 4.1 is complete.

4.1.3 Proof of Theorem 4.2

We still denote by E the Orlicz-Sobolev space $W_0^1 L_\Phi(\Omega)$. For any $\lambda > 0$ let J_λ be defined as in the above section of the section.

In order to prove Theorem 4.2 we apply to functional J_λ a symmetric version of the mountain pass lemma, recently developed by Kajikia in [41]. Before presenting the result in [41] we remember the following definition.

Definition 1. Let X be a real Banach space. We say that a subset A of X is *symmetric* if $u \in A$ implies $-u \in A$. For a closed symmetric set A which does not contain the origin, we define the *genus* $\gamma(A)$ of A as the smallest integer k such that there exists an odd continuous mapping from A to $\mathbf{R}^k \setminus \{0\}$. If there does not exist such an integer k, we define $\gamma(A) = +\infty$. Moreover, we set $\gamma(\emptyset) = 0$. Finally, we denote by Γ_k the family

$$\Gamma_k = \{A \subset X;\ 0 \notin A \text{ and } \gamma(A) \geq k\}.$$

We state now the symmetric mountain pass lemma of Kajikia (see Theorem 1 in [41]).

Theorem 4.3. *Assume X is an infinite dimensional Banach space and $\Lambda \in C^1(X, \mathbf{R})$ satisfies conditions (A1) and (A2) below.*

(A1) $\Lambda(u)$ is even, bounded from below, $\Lambda(0) = 0$ and $\Lambda(u)$ satisfies the Palais-Smale condition (i.e., any sequence $\{u_n\}$ in X such that $\{\Lambda(u_n)\}$ is bounded and $\Lambda^{'}(u_n) \to 0$ in $X^\star$ as $n \to \infty$ has a convergent subsequence);

(A2) For each $k \in \mathbf{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u\in A_k} \Lambda(u) < 0$.

Under the above assumptions, either (i) or (ii) below hold true.

(i) There exists a sequence $\{u_n\}$ *such that* $\Lambda^{'}(u_n) = 0$, $\Lambda(u_n) < 0$ *and* $\{u_n\}$ *converges to zero;*

(ii) There exist two sequences $\{u_n\}$ *and* $\{v_n\}$ *such that* $\Lambda^{'}(u_n) = 0$, $\Lambda(u_n) = 0$, $u_n \neq 0$, $\lim_{n\to\infty} u_n = 0$, $\Lambda^{'}(v_n) = 0$, $\Lambda(v_n) = 0$, *and* v_n *converges to a non-zero limit.*

In order to apply Theorem 4.3 to functional J_λ we prove two auxiliary results.

Lemma 4.3. *Functional* J_λ *satisfies condition (A1) from Theorem 4.3.*

Proof. Clearly, $J_\lambda(u) = J_\lambda(-u)$ for any $u \in E$, i.e. J_λ is even, and $J_\lambda(0) = 0$. On the other hand, since by relation (4.7) we have

$$\int_\Omega \Phi(|\nabla u(x)|)\, dx \geq \|u\|_{0,\Phi}^{(p)^0}, \quad \forall\, u \in E \text{ with } \|u\|_{0,\Phi} < 1,$$

while by Lemma C.9 in [19] we have

$$\int_\Omega \Phi(|\nabla u(x)|)\, dx \geq \|u\|_{0,\Phi}^{(p)_0}, \quad \forall\, u \in E \text{ with } \|u\|_{0,\Phi} > 1,$$

we deduce that

$$\int_\Omega \Phi(|\nabla u(x)|)\, dx \geq \alpha(\|u\|_{0,\Phi}), \quad \forall\, u \in E, \tag{4.16}$$

where $\alpha : [0,\infty) \to \mathbf{R}$, $\alpha(t) = t^{(p)^0}$ if $t < 0$ and $\alpha(t) = t^{(p)_0}$ if $t > 1$.

By Remark 1, space E is continuously embedded in $L^{q^\pm}(\Omega)$. Thus, there exist two positive constants d_1 and d_2 such that

$$\int_\Omega |u|^{q^+}\, dx \leq d_1 \|u\|_{0,\Phi}^{q^+}, \quad \int_\Omega |u|^{q^-}\, dx \leq d_2 \|u\|_{0,\Phi}^{q^-}, \quad \forall\, u \in E. \tag{4.17}$$

Combining relations (4.16) and (4.17) we get

$$J_\lambda(u) \geq \alpha(\|u\|_{0,\Phi}) - \frac{d_1\lambda}{q^-}\|u\|_{0,\Phi}^{q^+} - \frac{d_2\lambda}{q^-}\|u\|_{0,\Phi}^{q^-}, \quad \forall\, u \in E.$$

Since by relation (4.5) we have $q^+ < (p)_0$ the above relation shows that J_λ is bounded from below.

Next, we show that J_λ satisfies the Palais-Smale condition. Let $\{u_n\}$ be a sequence in E such that $\{J_\lambda(u_n)\}$ is bounded and $J^{'}(u_n) \to 0$ in $E^{'}$, as $n \to \infty$. We show that $\{u_n\}$ is bounded in E. Assume by contradiction the contrary. Then, passing eventually to a subsequence, still denoted by $\{u_n\}$, we may assume that $\|u_n\|_{0,\Phi} \to \infty$ as $n \to \infty$. Thus we may consider that $\|u_n\|_{0,\Phi} > 1$ for any integer n.

Following our assumptions, there is a positive constant M such that for all n large enough we have

$$\begin{aligned}
M+1+\|u_n\|_{0,\Phi} &\geq J_\lambda(u_n)-\frac{1}{q^-}\langle J^{'}(u_n),u_n\rangle \\
&= \int_\Omega \Phi(|\nabla u_n|)\,dx-\lambda\int_\Omega\frac{1}{q(x)}|u_n|^{q(x)}\,dx-\frac{1}{q^-}\cdot\int_\Omega\varphi(|\nabla u_n(x)|)|\nabla u_n(x)|\,dx+ \\
&\quad \frac{\lambda}{q^-}\int_\Omega|u_n|^{q(x)}\,dx \\
&\geq \int_\Omega \Phi(|\nabla u_n|)\,dx-\frac{1}{q^-}\cdot\int_\Omega\varphi(|\nabla u_n(x)|)|\nabla u_n(x)|\,dx \\
&\geq \left(1-\frac{(p)^0}{q^-}\right)\int_\Omega\Phi(|\nabla u_n|)\,dx \\
&\geq \left(1-\frac{(p)^0}{q^-}\right)\|u_n\|_{0,\Phi}^{(p)_0}.
\end{aligned}$$

Since $(p)_0>1$, letting $n\to\infty$ we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E. Similar arguments as those used in the end of the proof of Theorem 4.1 imply that, up to a subsequence, $\{u_n\}$ converges strongly in E.

The proof of Lemma 4.3 is complete.

Lemma 4.4. *Functional J_λ satisfies condition (A2) from Theorem 4.3.*

Proof. We construct a sequence of subsets $A_k\in\Gamma_k$ such that $\sup_{u\in A_k}J_\lambda(u)<0$, for each $k\in\mathbf{N}$.

Let $x_1\in\Omega$ and $r_1>0$ be such that $\overline{B_{r_1}(x_1)}\subset\Omega$ and $|\overline{B_{r_1}(x_1)}|<|\Omega|/2$. Consider $\theta_1\in C_0^\infty(\Omega)$, a function with $\text{supp}(\theta_1)=\overline{B_{r_1}(x_1)}$.

Define $\Omega_1=\Omega\setminus\overline{B_{r_1}(x_1)}$.

Next, let $x_2\in\Omega$ and $r_2>0$ be such that $\overline{B_{r_2}(x_2)}\subset\Omega_1$ and $|\overline{B_{r_2}(x_2)}|<|\Omega_1|/2$. Consider $\theta_2\in C_0^\infty(\Omega)$, a function with $\text{supp}(\theta_2)=\overline{B_{r_2}(x_2)}$.

Continuing the process described above we can construct by recurrence a sequence of functions θ_1, θ_2,..., $\theta_k\in C_0^\infty(\Omega)$ such that $\text{supp}(\theta_i)\neq\text{supp}(\theta_j)$ if $i\neq j$ and $|\text{supp}(\theta_i)|>0$ for any $i,\ j\in\{1,...,k\}$.

We define the finite dimensional subspace of E,

$$F=\text{span}\{\theta_1,\theta_2,...,\theta_k\}.$$

Clearly, $\dim F=k$ and $\int_\Omega|\theta|^{q(x)}\,dx>0$, for any $\theta\in F\setminus\{0\}$. We denote by S_1 the unit sphere in E, i.e. $S_1=\{u\in E;\ \|u\|_{0,\Phi}=1\}$. For any number $t\in(0,1)$ we define the set

$$A_k(t)=t\cdot(S_1\cap F).$$

Since for any bounded symmetric neighborhood ω of the origin in $\mathbf{R}^k$, $\gamma(\partial\omega)=k$ holds (see Proposition 5.2 in [75]) we deduce that $\gamma(A_k(t))=k$ for any $t\in(0,1)$.

Finally, we show that for each integer k there exists $t_k\in(0,1)$ such that

$$\sup_{u\in A_k(t_k)}J_\lambda(u)<0.$$

For any $t \in (0,1)$ we have

$$\begin{aligned}
\sup_{u\in A_k(t)} J_\lambda(u) &\leq \sup_{\theta\in S_1\cap F} J_\lambda(t\theta)\\
&= \sup_{\theta\in S_1\cap F}\left\{\int_\Omega \Phi(t|\nabla\theta|)\,dx - \lambda\int_\Omega \frac{1}{q(x)} t^{q(x)}|\theta|^{q(x)}\,dx\right\}\\
&\leq \sup_{\theta\in S_1\cap F}\left\{t^{(p)_0}\int_\Omega \Phi(|\nabla\theta|)\,dx - \frac{\lambda t^{q^+}}{q^+}\int_\Omega |\theta|^{q(x)}\,dx\right\}\\
&= \sup_{\theta\in S_1\cap F}\left\{t^{(p)_0}\left(1-\frac{\lambda}{q^+}\cdot\frac{1}{t^{(p)_0-q^+}}\cdot\int_\Omega|\theta|^{q(x)}\,dx\right)\right\}.
\end{aligned}$$

Since $S_1\cap F$ is compact we have $m=\min_{\theta\in S_1\cap F}\int_\Omega|\theta|^{q(x)}\,dx>0$. Combining that fact with the information given by relation (4.5), that is $(p)_0>q^+$, we deduce that we can choose $t_k\in(0,1)$ small enough such that

$$1-\frac{\lambda}{q^+}\cdot\frac{1}{t^{(p)_0-q^+}}\cdot m<0.$$

The above relations yield

$$\sup_{u\in A_k(t_k)} J_\lambda(u)<0.$$

The proof of Lemma 4.4 is complete.

Proof of Theorem 4.2. Using Lemmas 4.3 and 4.4 we deduce that we can apply Theorem 4.3 to functional J_λ. So, there exists a sequence $\{u_n\}\subset E$ such that $J^{'}(u_n)=0$, for each n, $J_\lambda(u_n)\leq 0$ and $\{u_n\}$ converges to zero in E.

The proof of Theorem 4.2 is complete.

4.1.4 Examples

Next, we point out two concrete examples of problems to which we can apply the main results of this section.

Example 1. We consider problem

$$\begin{cases} -\text{div}(\log(1+|\nabla u|^r)|\nabla u|^{p-2}\nabla u)=\lambda|u|^{q(x)-2}u, & \text{for} \quad x\in\Omega\\ u=0, & \text{for} \quad x\in\partial\Omega,\end{cases} \tag{4.18}$$

where p and r are real numbers such that $1<p$, r, $N>p+r$ and $q(x)$ is a continuous function on $\overline{\Omega}$ such that $1<q(x)$ for all $x\in\overline{\Omega}$ and, furthermore,

$$\inf_\Omega q(x)<p \quad \text{and} \quad \sup_\Omega q(x)<\frac{Np}{N-p}.$$

In this case we have

$$\varphi(t)=\log(1+|t|^r)\cdot|t|^{p-2}t, \qquad \text{for all } t\in\mathbf{R}$$

and

$$\Phi(t) = \int_0^t \varphi(s)\, ds, \qquad \text{for all } t \in \mathbf{R}.$$

Clearly, φ is an odd, increasing homeomorphism of $\mathbf{R}$ into $\mathbf{R}$, while Φ is convex and even on $\mathbf{R}$ and increasing from $\mathbf{R}_+$ to $\mathbf{R}_+$.

By Example 2 on p. 243 in [19] we know that

$$(p)_0 = p \quad \text{and} \quad (p)^0 = p + r$$

and thus relation (4.3) in Theorem 4.1 is satisfied. It is easy to deduce that relations (4.2) and (4.4) are fulfilled. Thus, we have verified that we can apply Theorem 4.1 in order to find out that there exists $\lambda^\star > 0$ such that any $\lambda \in (0, \lambda^\star)$ is an eigenvalue of problem (4.18).

Example 2. We consider problem

$$\begin{cases} -\text{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{\log(1+|\nabla u|)}\right) = \lambda |u|^{q(x)-2}u, & \text{for} \quad x \in \Omega \\ u = 0, & \text{for} \quad x \in \partial\Omega\,, \end{cases} \tag{4.19}$$

where p is a real number such that $p > N+1$ and $q \in C(\overline{\Omega})$ satisfies $1 < q(x) < p-1$ for any $x \in \overline{\Omega}$. In this case we have

$$\varphi(t) = \frac{|t|^{p-2}}{\log(1+|t|)} t$$

and

$$\Phi(t) = \int_0^t \varphi(s)\, ds,$$

is an increasing continuous function from $\mathbf{R}^+$ to $\mathbf{R}^+$, with $\Phi(0) = 0$ and such that function $\Phi(\sqrt{t})$ is convex. By Example 3 on p. 243 in [19] we have

$$(p)_0 = p - 1 < (p)^0 = p = \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}.$$

Thus, conditions (4.2), (4.5) and (4.6) from Theorem 4.2 and Remark 2 are verified. We deduce that every $\lambda > 0$ is an eigenvalue of problem (4.19). Moreover, for each $\lambda > 0$ there exists a sequence of eigenvectors $\{u_n\}$ such that $\lim_{n\to\infty} u_n = 0$ in $W_0^1 L_\Phi(\Omega)$.

4.2 Eigenvalue problem $-\text{div}((a_1(|\nabla u|) + a_2(|\nabla u|))\nabla u) = \lambda |u|^{q(x)-2}u$

4.2.1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^N$ ($N \geq 3$) with smooth boundary $\partial\Omega$. In this section we are concerned with the study of the eigenvalue problem

$$\begin{cases} -\text{div}((a_1(|\nabla u|) + a_2(|\nabla u|))\nabla u) = \lambda |u|^{q(x)-2}u, & \text{for} \quad x \in \Omega \\ u = 0, & \text{for} \quad x \in \partial\Omega\,. \end{cases} \tag{4.20}$$

We assume that functions $a_i : (0, \infty) \to \mathbf{R}$, $i = 1, 2$, are such that mappings $\varphi_i : \mathbf{R} \to \mathbf{R}$, $i = 1, 2$, defined by

$$\varphi_i(t) = \begin{cases} a_i(|t|)t, & \text{for} \quad t \neq 0 \\ 0, & \text{for} \quad t = 0\,, \end{cases}$$

are odd, increasing homeomorphisms from $\mathbf{R}$ onto $\mathbf{R}$. We also suppose throughout this section that $\lambda > 0$ and $q : \overline{\Omega} \to (0, \infty)$ is a continuous function.

Since the operator in the divergence form is nonhomogeneous we introduce an Orlicz-Sobolev space setting for problems of this type (see Chapter 1 for definitions, notations and properties of Orlicz-Sobolev spaces). We introduce spaces $W_0^1 L_{\Phi_1}(\Omega)$ and $W_0^1 L_{\Phi_2}(\Omega)$, where

$$\Phi_i(t) = \int_0^t \varphi_i(s)\, ds, \qquad \text{for all } t \in \mathbf{R},\ i = 1, 2\,.$$

We define

$$(p_i)_0 := \inf_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)} \ \text{ and } \ (p_i)^0 := \sup_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)},\ i = 1, 2\,,$$

and assume that condition (1.10) is fulfilled for $i = 1, 2$ and

$$\lim_{t \to 0} \int_t^1 \frac{(\Phi_i)^{-1}(s)}{s^{(N+1)/N}}\, ds < \infty, \quad \text{and} \quad \lim_{t \to \infty} \int_1^t \frac{(\Phi_i)^{-1}(s)}{s^{(N+1)/N}}\, ds = \infty,\ i = 1, 2\,. \tag{4.21}$$

We introduce the Orlicz-Sobolev conjugate $(\Phi_i)_\star$ of Φ_i, $i = 1, 2$, defined as

$$(\Phi_i)_\star^{-1}(t) = \int_0^t \frac{(\Phi_i)^{-1}(s)}{s^{(N+1)/N}}\, ds.$$

We will analyze problem (4.20) under the following assumptions

$$1 < (p_2)_0 \le (p_2)^0 < q(x) < (p_1)_0 \le (p_1)^0, \quad \forall\, x \in \overline{\Omega} \tag{4.22}$$

and

$$\lim_{t \to \infty} \frac{|t|^{q^+}}{(\Phi_2)_\star(kt)} = 0, \quad \text{for all } k > 0\,. \tag{4.23}$$

4.2.2 Auxiliary results

In this section we point out certain useful results of great interest.

Lemma 4.5. *The following relations hold true*

$$\int_\Omega \Phi_i(|\nabla u(x)|)\, dx \le \|u\|_{0,\Phi_i}^{(p_i)_0}, \quad \forall\, u \in W_0^1 L_{\Phi_i}(\Omega) \text{ with } \|u\|_{0,\Phi_i} < 1,\ i = 1, 2\,;$$

$$\int_\Omega \Phi_i(|\nabla u(x)|)\, dx \ge \|u\|_{0,\Phi_i}^{(p_i)_0}, \quad \forall\, u \in W_0^1 L_{\Phi_i}(\Omega) \text{ with } \|u\|_{0,\Phi_i} > 1,\ i = 1, 2\,;$$

$$\int_\Omega \Phi_i(|\nabla u(x)|)\, dx \ge \|u\|_{0,\Phi_i}^{(p_i)^0}, \quad \forall\, u \in W_0^1 L_{\Phi_i}(\Omega) \text{ with } \|u\|_{0,\Phi_i} < 1,\ i = 1, 2\,;$$

$$\int_\Omega \Phi_i(|\nabla u(x)|)\, dx \le \|u\|_{0,\Phi_i}^{(p_i)^0}, \quad \forall\, u \in W_0^1 L_{\Phi_i}(\Omega) \text{ with } \|u\|_{0,\Phi_i} > 1,\ i = 1, 2\,.$$

Proof. The proof of the first two inequalities can be carried out as in [19, Lemma C.9].

Next, assume $\|u\|_{0,\Phi_i} < 1$. Let $\xi \in (0, \|u\|_{0,\Phi_i})$. By the definition of $(p_i)^0$, it is easy to prove that

$$\Phi_i(t) \geq \tau^{(p_i)^0} \Phi_i(t/\tau), \quad \forall\, t > 0,\ \tau \in (0,1).$$

Using the above relation we have

$$\int_\Omega \Phi_i(|\nabla u(x)|)\, dx \geq \xi^{(p_i)^0} \cdot \int_\Omega \Phi_i \left(\frac{|\nabla u(x)|}{\xi} \right)\, dx. \tag{4.24}$$

Defining $v(x) = u(x)/\xi$, for all $x \in \Omega$, we have $\|v\|_{0,\Phi_i} = \|u\|_{0,\Phi_i}/\xi > 1$. Using the first inequality of this lemma we find

$$\int_\Omega \Phi_i(|\nabla v(x)|)\, dx \geq \|v\|_{0,\Phi_i}^{(p_i)_0} > 1. \tag{4.25}$$

Relations (4.24) and (4.25) show that

$$\int_\Omega \Phi_i(|\nabla u(x)|)\, dx \geq \xi^{(p_i)^0}.$$

Letting $\xi \nearrow \|u\|_{0,\Phi_i}$ in the above inequality we obtain

$$\int_\Omega \Phi_i(|\nabla u(x)|)\, dx \geq \|u\|_{0,\Phi_i}^{{p_i}^0}, \quad \forall\, u \in W_0^1 L_{\Phi_i}(\Omega) \text{ with } \|u\|_{0,\Phi_i} < 1.$$

Finally, we prove the last inequality in the lemma. It is easy to show that

$$\frac{\Phi_i(\sigma t)}{\Phi_i(t)} \leq \sigma^{{p_i}^0}, \quad \forall\, t > 0 \text{ and } \sigma > 1. \tag{4.26}$$

Then, for all $u \in W_0^1 L_{\Phi_i}(\Omega)$ with $\|u\|_{0,\Phi_i} > 1$, relation (4.26) implies

$$\begin{aligned} \int_\Omega \Phi_i(|\nabla u(x)|)\, dx &= \int_\Omega \Phi_i \left(\|u\|_{0,\Phi_i} \frac{|\nabla u(x)|}{\|u\|_{0,\Phi_i}} \right) dx \\ &\leq \|u\|_{0,\Phi_i}^{(p_i)^0} \int_\Omega \Phi_i \left(\frac{|\nabla u(x)|}{\|u\|_{0,\Phi_i}} \right) dx \\ &\leq \|u\|_{0,\Phi_i}^{(p_i)^0}. \end{aligned}$$

The proof of Lemma 4.5 is complete.

Lemma 4.6. *Assume relation (4.22) holds true. Then the continuous embedding*

$$W_0^1 L_{\Phi_1}(\Omega) \subset W_0^1 L_{\Phi_2}(\Omega)$$

holds true.

Proof. By [3, Lemma 8.12 (b)] it is enough to show that Φ_1 dominates Φ_2 near infinity, i.e. there exist $k > 0$ and $t_0 > 0$ such that

$$\Phi_2(t) \leq \Phi_1(k \cdot t), \quad \forall\, t \geq t_0\,.$$

Indeed, since by (4.22) we have $(p_2)^0 < (p_1)_0$ it follows that

$$\frac{\varphi_2(t)}{\Phi_2(t)} < \frac{\varphi_1(t)}{\Phi_1(t)}, \quad \forall\, t > 0\,.$$

The above relation and some elementary computations imply

$$\left(\frac{\Phi_1(t)}{\Phi_2(t)}\right)' > 0, \quad \forall\, t > 0\,.$$

Thus, we deduce that $\Phi_1(t)/\Phi_2(t)$ is increasing for any $t \in (0,\infty)$. It follows that for a fixed $t_0 \in (0,\infty)$ we have

$$\frac{\Phi_1(t_0)}{\Phi_2(t_0)} < \frac{\Phi_1(t)}{\Phi_2(t)}, \quad \forall\, t > t_0\,.$$

Let $k \in (0, \min\{1, \Phi_1(t_0)/\Phi_2(t_0)\})$ be fixed. The above relations yield

$$\Phi_2(t) < \frac{1}{k} \cdot \Phi_1(t), \quad \forall\, t > t_0\,.$$

Finally, we point out that in order to end the proof of the lemma it is enough to show

$$\frac{1}{k} \cdot \Phi_1(t) \leq \Phi_1\left(\frac{1}{k} \cdot t\right), \quad \forall\, t > 0\,.$$

Indeed, define function $H : [0,\infty) \to \mathbf{R}$ by

$$H(t) = \Phi_1\left(\frac{1}{k} \cdot t\right) - \frac{1}{k} \cdot \Phi_1(t)\,.$$

Then we get

$$H^{'}(t) = \frac{1}{k} \cdot \left(\varphi_1\left(\frac{1}{k} \cdot t\right) - \varphi_1(t)\right)\,.$$

Since φ_1 is an increasing function and $1/k > 1$ we deduce that H is an increasing function. That fact combined with the remark that $H(0) = 0$ implies

$$H(t) \geq H(0) = 0, \quad \forall\, t \geq 0\,,$$

or

$$\frac{1}{k} \cdot \Phi_1(t) \leq \Phi_1\left(\frac{1}{k} \cdot t\right), \quad \forall\, t > 0\,.$$

The proof of Lemma 4.6 is complete.

Lemma 4.7. *Assume relation (4.22) holds true. Then there exists $c > 0$ such that the following inequality holds true*

$$c \cdot [\Phi_1(t) + \Phi_2(t)] \geq t^{(p_1)_0} + t^{(p_2)^0}, \quad \forall\, t \geq 0\,.$$

Proof. Using the definition of $(p_1)_0$ we deduce that

$$\left(\frac{\Phi_1(t)}{t^{(p_1)_0}}\right)' > 0, \quad \forall\, t > 0\,,$$

or, function $\Phi_1(t)/t^{(p_1)_0}$ for $t \in (0,\infty)$. Thus, we deduce that

$$\Phi_1(t) \geq \Phi_1(1) \cdot t^{(p_1)_0}, \quad \forall\, t > 1\,,$$

or letting $c_1 = 1/\Phi_1(1)$

$$c_1 \cdot \Phi_1(t) \geq t^{(p_1)_0}, \quad \forall\, t > 1\,. \tag{4.27}$$

Next, by the definition of $(p_2)^0$, it is easy to prove that

$$\Phi_2(t) \geq \tau^{(p_2)^0}\Phi_2(t/\tau), \quad \forall\, t > 0,\ \tau \in (0,1).$$

Letting $t \in (0,1)$ and $\tau = t$ the above inequality implies

$$\Phi_2(t) \geq t^{(p_2)^0} \cdot \Phi_2(1), \quad \forall\, t \in (0,1)\,,$$

or letting $c_2 = 1/\Phi_2(1)$

$$c_2 \cdot \Phi_2(t) \geq t^{(p_2)^0}, \quad \forall\, t \in (0,1)\,. \tag{4.28}$$

Finally, let $c = 2 \cdot \max\{c_1, c_2\}$. Then, since by relation (4.22) we have $(p_2)^0 < (p_1)_0$ and since relations (4.27) and (4.28) hold true we deduce that

$$c \cdot [\Phi_1(t) + \Phi_2(t)] \geq 2 \cdot t^{(p_1)_0} \geq t^{(p_1)_0} + t^{(p_2)^0}, \quad \forall\, t \geq 1\,,$$

and

$$c \cdot [\Phi_1(t) + \Phi_2(t)] \geq 2 \cdot t^{(p_2)^0} \geq t^{(p_1)_0} + t^{(p_2)^0}, \quad \forall\, t \in (0,1)\,.$$

The proof of Lemma 4.7 is complete.

4.2.3 Main result

Since we study problem (4.20) under hypothesis (4.22) by Lemma 4.6 it follows that $W_0^1 L_{\Phi_1}(\Omega)$ is continuously embedded in $W_0^1 L_{\Phi_2}(\Omega)$. Thus, a solution for a problem of type (4.20) will be sought in the variable exponent space $W_0^1 L_{\Phi_1}(\Omega)$.

We say that $\lambda \in \mathbf{R}$ is an *eigenvalue* of problem (4.20) if there exists $u \in W_0^1 L_{\Phi_1}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega (a_1(|\nabla u|) + a_2(|\nabla u|))\nabla u \nabla v\, dx - \lambda \int_\Omega |u|^{q(x)-2} uv\, dx = 0\,,$$

for all $v \in W_0^1 L_{\Phi_1}(\Omega)$. We point out that if λ is an eigenvalue of problem (4.20) then the corresponding $u \in W_0^1 L_{\Phi_1}(\Omega) \setminus \{0\}$ is a *weak solution* of (4.20).

Define

$$\lambda_1 := \inf_{u \in W_0^1 L_{\Phi_1}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \Phi_1(|\nabla u|)\, dx + \int_\Omega \Phi_2(|\nabla u|)\, dx}{\displaystyle\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\, dx}\,.$$

Our main result is given by the following theorem.

Theorem 4.4. *Assume that conditions (4.21), (4.22) and (4.23) are fulfilled. Then* $\lambda_1 > 0$. *Moreover, any* $\lambda \in [\lambda_1, \infty)$ *is an eigenvalue of problem (4.20). Furthermore, there exists a positive constant* λ_0 *such that* $\lambda_0 \leq \lambda_1$ *and any* $\lambda \in (0, \lambda_0)$ *is not an eigenvalue of problem (4.20).*

Remark 4.1. *Relations (4.21) and (4.23) enable us to apply [33, Theorem 2.2] (see also [3, Theorem 8.33]) in order to obtain that* $W_0^1 L_{\Phi_2}(\Omega)$ *is compactly embedded in* $L^{q^+}(\Omega)$. *That fact combined with the continuous embedding of* $L^{q^+}(\Omega)$ *in* $L^{p(\cdot)}(\Omega)$ *and with the result of Lemma 4.6 assures that* $W_0^1 L_{\Phi_1}(\Omega)$ *is compactly embedded in* $L^{p(\cdot)}(\Omega)$.

4.2.4 Proof of main result

Let E denote the generalized Sobolev space $W_0^1 L_{\Phi_1}(\Omega)$. In this section we denote by $\|\cdot\|_{0,\Phi_1}$ the norm on $W_0^1 L_{\Phi_1}(\Omega)$ and by $\|\cdot\|_{0,\Phi_2}$ the norm on $W_0^1 L_{\Phi_2}(\Omega)$.

In order to prove our main result we introduce four functionals J, I, J_1, $I_1 : E \to \mathbf{R}$ by

$$J(u) = \int_\Omega \Phi_1(|\nabla u|)\, dx + \int_\Omega \Phi_2(|\nabla u|)\, dx,$$

$$I(u) = \int_\Omega \frac{1}{q(x)} |u|^{q(x)}\, dx,$$

$$J_1(u) = \int_\Omega a_1(|\nabla u|)|\nabla u|^2\, dx + \int_\Omega a_2(|\nabla u|)|\nabla u|^2\, dx,$$

$$I_1(u) = \int_\Omega |u|^{q(x)}\, dx.$$

Standard arguments imply that $J,\ I \in C^1(E, \mathbf{R})$ and

$$\langle J'(u), v\rangle = \int_\Omega (a_1(|\nabla u|) + a_2(|\nabla u|))\nabla u \nabla v\, dx,$$

$$\langle I'(u), v\rangle = \int_\Omega |u|^{q(x)-2} uv\, dx$$

for all $u,\ v \in E$. We will prove Theorem 4.4 in four steps.

• Step 1. We show that $\lambda_1 > 0$.

By Lemma 4.7 and relation (4.22) we deduce that the following relations hold true

$$2 \cdot c \cdot (\Phi_1(|\nabla u(x)|) + \Phi_2(|\nabla u(x)|)) \geq 2 \cdot (|\nabla u(x)|^{(p_1)_0} + |\nabla u(x)|^{(p_2)^0}) \geq |\nabla u(x)|^{q^+} + |\nabla u(x)|^{q^-}$$

and

$$|u(x)|^{q^+} + |u(x)|^{q^-} \geq |u(x)|^{q(x)}.$$

Integrating the above inequalities we find

$$2c \cdot \int_\Omega (\Phi_1(|\nabla u(x)|) + \Phi_2(|\nabla u(x)|))\, dx \geq \int_\Omega (|\nabla u|^{q^+} + |\nabla u|^{q^-})\, dx, \quad \forall\, u \in E \tag{4.29}$$

and

$$\int_\Omega (|u|^{q^+} + |u|^{q^-})\, dx \geq \int_\Omega |u|^{q(x)}\, dx \quad \forall\, u \in E. \tag{4.30}$$

On the other hand, it is well known that there exist two positive constants λ_{q^+} and λ_{q^-} such that

$$\int_\Omega |\nabla u|^{q^+}\, dx \geq \lambda_{q^+} \int_\Omega |u|^{q^+}\, dx, \quad \forall\, u \in W_0^{1,q^+}(\Omega) \tag{4.31}$$

and

$$\int_\Omega |\nabla u|^{q^-}\, dx \geq \lambda_{q^-} \int_\Omega |u|^{q^-}\, dx, \quad \forall\, u \in W_0^{1,q^-}(\Omega). \tag{4.32}$$

Using again the fact that $q^- \leq q^+ < (p_1)_0$ and a similar technique as that used in the proof of Lemma 4.6 we deduce that E is continuously embedded in $W_0^{1,q^+}(\Omega)$ and in $W_0^{1,q^-}(\Omega)$. Thus, inequalities (4.31) and (4.32) hold true for any $u \in E$.

Using inequalities (4.31), (4.32) and (4.30) it is clear that there exists a positive constant μ such that

$$\int_\Omega (|\nabla u|^{q^+} + |\nabla u|^{q^-})\, dx \geq \mu \int_\Omega |u|^{q(x)}\, dx \quad \forall\, u \in E. \tag{4.33}$$

Next, inequalities (4.33) and (4.29) yield

$$\int_\Omega (\Phi_1(|\nabla u(x)|) + \Phi_2(|\nabla u(x)|))\, dx \geq \frac{\mu}{2c} \int_\Omega |u|^{q(x)}\, dx \quad \forall\, u \in E. \tag{4.34}$$

The above inequality implies

$$J(u) \geq \frac{\mu \cdot q^-}{2c} I(u) \quad \forall\, u \in E. \tag{4.35}$$

The last inequality assures that $\lambda_1 > 0$ and, thus, step 1 is verified.

Remark 4.2. *We point out that by the definitions of $(p_i)_0$, $i = 1, 2$, we have*

$$a_i(t) \cdot t^2 = \varphi_i(t) \cdot t \geq (p_i)_0 \Phi_i(t), \quad \forall\, t > 0\,.$$

The above inequality and relation (4.34) imply

$$\lambda_0 = \inf_{v \in E \setminus \{0\}} \frac{J_1(v)}{I_1(v)} > 0\,. \tag{4.36}$$

• Step 2. We show that λ_1 is an eigenvalue of problem (4.20).

In order to show that λ_1 is an eigenvalue of problem (4.20) we point out certain auxiliary results.

Lemma 4.8. *The following relations hold true:*

$$\lim_{\|u\| \to \infty} \frac{J(u)}{I(u)} = \infty \tag{4.37}$$

and

$$\lim_{\|u\| \to 0} \frac{J(u)}{I(u)} = \infty. \tag{4.38}$$

Proof. Since E is continuously embedded in $L^{q^\pm}(\Omega)$ it follows that there exist two positive constants c_1 and c_2 such that

$$\|u\|_{0,\Phi_1} \geq c_1 \cdot |u|_{q^+}, \quad \forall\, u \in E \tag{4.39}$$

and

$$\|u\|_{0,\Phi_1} \geq c_2 \cdot |u|_{q^-}, \quad \forall\, u \in E. \tag{4.40}$$

For any $u \in E$ with $\|u\|_{0,\Phi_1} > 1$ by Lemma 4.5 and relations (4.30), (4.39), (4.40) we infer

$$\frac{J(u)}{I(u)} \geq \frac{\|u\|_{0,\Phi_1}^{(p_1)_0}}{\dfrac{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}}{q^-}} \geq \frac{\dfrac{\|u\|_{0,\Phi_1}^{p_1^-}}{p_1^+}}{\dfrac{c_1^{-q^+}\|u\|_{0,\Phi_1}^{q^+} + c_2^{-q^-}\|u\|_{0,\Phi_1}^{q^-}}{q^-}}.$$

Since $(p_1)_0 > q^+ \geq q^-$, passing to the limit as $\|u\|_{0,\Phi_1} \to \infty$ in the above inequality we deduce that relation (4.37) holds true.

Next, by Lemma 4.6, space $W_0^1 L_{\Phi_1}(\Omega)$ is continuously embedded in $W_0^1 L_{\Phi_2}(\Omega)$. Thus, if $\|u\|_{0,\Phi_1} \to 0$ then $\|u\|_{0,\Phi_2} \to 0$.

The above remarks enable us to affirm that for any $u \in E$ with $\|u\|_{0,\Phi_1} < 1$ small enough we have $\|u\|_{0,\Phi_2} < 1$.

On the other hand, since (4.23) holds true we deduce that $W_0^1 L_{\Phi_2}(\Omega)$ is continuously embedded in $L^{q^\pm}(\Omega)$. It follows that there exist two positive constants d_1 and d_2 such that

$$\|u\|_{0,\Phi_2} \geq d_1 \cdot |u|_{q^+}, \quad \forall\, u \in W_0^1 L_{\Phi_2}(\Omega) \tag{4.41}$$

and

$$\|u\|_{0,\Phi_2} \geq d_2 \cdot |u|_{q^-}, \quad \forall\, u \in W_0^1 L_{\Phi_2}(\Omega). \tag{4.42}$$

Thus, for any $u \in E$ with $\|u\|_{0,\Phi_1} < 1$ small enough, Lemma 4.5 and relations (4.30), (4.41), (4.42) imply

$$\frac{J(u)}{I(u)} \geq \frac{\int_\Omega \Phi_2(|\nabla u|)\,dx}{\dfrac{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}}{q^-}} \geq \frac{\|u\|_{0,\Phi_2}^{(p_2)^0}}{\dfrac{d_1^{-q^+}\|u\|_{0,\Phi_2}^{q^+} + d_2^{-q^-}\|u\|_{0,\Phi_2}^{q^-}}{q^-}}.$$

Since $(p_2)^0 < q^- \leq q^+$, passing to the limit as $\|u\|_{0,\Phi_1} \to 0$ (and thus, $\|u\|_{0,\Phi_2} \to 0$) in the above inequality we deduce that relation (4.38) holds true.

The proof of Lemma 4.8 is complete.

Lemma 4.9. *There exists $u \in E \setminus \{0\}$ such that $\frac{J(u)}{I(u)} = \lambda_1$.*

Proof. Let $\{u_n\} \subset E \setminus \{0\}$ be a minimizing sequence for λ_1, i.e.

$$\lim_{n\to\infty} \frac{J(u_n)}{I(u_n)} = \lambda_1 > 0. \tag{4.43}$$

By relation (4.37) it is clear that $\{u_n\}$ is bounded in E. Since E is reflexive it follows that there exists $u \in E$ such that u_n converges weakly to u in E. On the other hand, it is easy to show that functional J is weakly lower semi-continuous. Thus, we find

$$\liminf_{n\to\infty} J(u_n) \geq J(u). \tag{4.44}$$

By Remark 4.1 it follows that E is compactly embedded in $L^{p(\cdot)}(\Omega)$. Thus, u_n converges strongly in $L^{p(\cdot)}(\Omega)$. Then, by relation (1.5) it follows that

$$\lim_{n\to\infty} I(u_n) = I(u). \tag{4.45}$$

Relations (4.44) and (4.45) imply that if $u \not\equiv 0$ then

$$\frac{J(u)}{I(u)} = \lambda_1.$$

Thus, in order to conclude that the lemma holds true it is enough to show that u can not be trivial. Assume by contradiction the contrary. Then u_n converges weakly to 0 in E and strongly in $L^{q(\cdot)}(\Omega)$. In other words, we will have

$$\lim_{n\to\infty} I(u_n) = 0. \tag{4.46}$$

Letting $\epsilon \in (0, \lambda_1)$ be fixed by relation (4.43) we deduce that for n large enough we have

$$|J(u_n) - \lambda_1 I(u_n)| < \epsilon I(u_n),$$

or

$$(\lambda_1 - \epsilon)I(u_n) < J(u_n) < (\lambda_1 + \epsilon)I(u_n).$$

Passing to the limit in the above inequalities and taking into account that relation (4.46) holds true we find

$$\lim_{n\to\infty} J(u_n) = 0.$$

That fact combined with the conclusion of Lemma 4.5 implies that actually u_n converges strongly to 0 in E, i.e. $\lim_{n\to\infty} \|u_n\|_{0,\Phi_1} = 0$. By this information and relation (4.38) we get

$$\lim_{n\to\infty} \frac{J(u_n)}{I(u_n)} = \infty,$$

and this is a contradiction. Thus, $u \not\equiv 0$.

The proof of Lemma 4.9 is complete.

By Lemma 4.9 we conclude that there exists $u \in E \setminus \{0\}$ such that

$$\frac{J(u)}{I(u)} = \lambda_1 = \inf_{w\in E\setminus\{0\}} \frac{J(w)}{I(w)}. \tag{4.47}$$

Then, for any $v \in E$ we have

$$\frac{d}{d\epsilon}\frac{J(u+\epsilon v)}{I(u+\epsilon v)}\,|_{\epsilon=0} = 0\ .$$

A simple computation yields

$$\int_\Omega (a_1(|\nabla u|)+a_2(|\nabla u|))\nabla u\nabla v\,dx\cdot I(u)-J(u)\cdot\int_\Omega |u|^{q(x)-2}uv\,dx=0,\quad \forall\, v\in E. \tag{4.48}$$

Relation (4.48) combined with the fact that $J(u)=\lambda_1 I(u)$ and $I(u)\neq 0$ implies the fact that λ_1 is an eigenvalue of problem (4.20). Thus, step 2 is verified.

• Step 3. We show that any $\lambda\in(\lambda_1,\infty)$ is an eigenvalue of problem (4.20).

Let $\lambda\in(\lambda_1,\infty)$ be arbitrary but fixed. Define $T_\lambda : E\to\mathbf{R}$ by

$$T_\lambda(u)=J(u)-\lambda I(u).$$

Clearly, $T_\lambda\in C^1(E,\mathbf{R})$ with

$$\langle T_\lambda^{'}(u),v\rangle=\langle J^{'}(u),v\rangle-\lambda\langle I^{'}(u),v\rangle,\quad \forall\, u\in E.$$

Thus, λ is an eigenvalue of problem (4.20) if and only if there exists $u_\lambda\in E\setminus\{0\}$ a critical point of T_λ.

With similar arguments as in the proof of relation (4.37) we can show that T_λ is coercive, i.e. $\lim_{\|u\|\to\infty}T_\lambda(u)=\infty$. On the other hand, it is known that functional T_λ is weakly lower semi-continuous. These two facts enable us to apply [75, Theorem 1.2] in order to prove that there exists $u_\lambda\in E$ a global minimum point of T_λ and, thus, a critical point of T_λ. In order to conclude that step 4 holds true it is enough to show that u_λ is not trivial. Indeed, since $\lambda_1=\inf_{u\in E\setminus\{0\}}\frac{J(u)}{I(u)}$ and $\lambda>\lambda_1$ it follows that there exists $v_\lambda\in E$ such that

$$J(v_\lambda)<\lambda I(v_\lambda),$$

or

$$T_\lambda(v_\lambda)<0.$$

Thus,

$$\inf_E T_\lambda<0$$

and we conclude that u_λ is a nontrivial critical point of T_λ, or λ is an eigenvalue of problem (4.20). Thus, step 3 is verified.

• Step 4. We show that any $\lambda\in(0,\lambda_0)$, where λ_0 is given by relation (4.36), is not an eigenvalue of problem (4.20).

Indeed, assuming by contradiction that there exists $\lambda\in(0,\lambda_0)$ an eigenvalue of problem (4.20) it follows that there exists $u_\lambda\in E\setminus\{0\}$ such that

$$\langle J^{'}(u_\lambda),v\rangle=\lambda\langle I^{'}(u_\lambda),v\rangle,\quad \forall\, v\in E.$$

Thus, for $v=u_\lambda$ we find

$$\langle J^{'}(u_\lambda),u_\lambda\rangle=\lambda\langle I^{'}(u_\lambda),u_\lambda\rangle,$$

or

$$J_1(u_\lambda) = \lambda I_1(u_\lambda).$$

The fact that $u_\lambda \in E \setminus \{0\}$ assures that $I_1(u_\lambda) > 0$. Since $\lambda < \lambda_0$, the above information implies

$$J_1(u_\lambda) \geq \lambda_0 I_1(u_\lambda) > \lambda I_1(u_\lambda) = J_1(u_\lambda).$$

Clearly, the above inequalities lead to a contradiction. Thus, step 4 is verified.

By steps 2, 3 and 4 we deduce that $\lambda_0 \leq \lambda_1$.

The proof of Theorem 4.4 is now complete.

Remark 4.3. *We point out that by the proof of Theorem 4.4 we can not conclude if $\lambda_0 = \lambda_1$ or $\lambda_0 < \lambda_1$. Such a study rests open. In the case when it is possible to have $\lambda_0 < \lambda_1$, if such a case could arise, the question regarding the existence of an eigenvalue of problem (4.20) in the interval $[\lambda_0, \lambda_1)$ also rests an open problem.*

4.3 An optimization result

Let Ω be a bounded domain in $\mathbf{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$. Assume that $a_i : (0,\infty) \to \mathbf{R}$, $i = 1,2$, are two functions such that mappings $\varphi_i : \mathbf{R} \to \mathbf{R}$, $i = 1,2$, defined by

$$\varphi_i(t) = \begin{cases} a_i(|t|)t, & \text{for} \quad t \neq 0 \\ 0, & \text{for} \quad t = 0\,, \end{cases}$$

are odd, increasing homeomorphisms from $\mathbf{R}$ onto $\mathbf{R}$, λ is a real number, $V(x)$ is a potential and q_1, q_2, $m : \overline{\Omega} \to (1,\infty)$ are continuous functions. Let $f(x,u) = (|u|^{q_1(x)-2} + |u|^{q_2(x)-2})u$. We analyze the eigenvalue problem

$$\begin{cases} -\text{div}((a_1(|\nabla u|) + a_2(|\nabla u|))\nabla u) + V(x)|u|^{m(x)-2}u = \lambda f(x,u), & x \in \Omega \\ u = 0, & x \in \partial\Omega \quad . \end{cases} \tag{4.49}$$

Problem (4.49) can be placed in the context of the eigenvalue problem studied in the above section since in the particular case when $q_1(x) = q_2(x) = q(x)$ for any $x \in \overline{\Omega}$ and $V \equiv 0$ in Ω it becomes problem (4.20). The form of problem (4.49) becomes a natural extension of problem (4.20) with the presence of the potential V in the left-hand side of the equation and by considering that in the right-hand side we have $q_1 \neq q_2$ on $\overline{\Omega}$.

For $i = 1,2$ define

$$\Phi_i(t) = \int_0^t \varphi_i(s)\,ds, \qquad \text{for all } t \in \mathbf{R},\ i = 1,2\,.$$

Define function spaces $W_0^1 L_{\Phi_i}(\Omega)$ as in Chapter 1. We also refer to Chapter 1 for definitions, notations and properties of Orlicz-Sobolev spaces. Here we assume that conditions (1.10) and (1.12) are satisfied for $i = 1,2$.

We will study problem (4.49) when q_1, q_2, $m : \overline{\Omega} \to (1, \infty)$ are continuous functions satisfying the following assumptions:

$$1 < (p_2)_0 \leq (p_2)^0 < q_2^- \leq q_2^+ \leq m^- \leq m^+ \leq q_1^- \leq q_1^+ < (p_1)_0 \leq (p_1)^0 < N\,, \tag{4.50}$$

$$q_1^+ < [(p_2)_0]^\star := \frac{N(p_2)_0}{N-(p_2)_0}, \quad \forall\, x \in \overline{\Omega}\,, \tag{4.51}$$

and the potential $V : \Omega \to \mathbf{R}$ satisfies

$$V \in L^{r(\cdot)}(\Omega), \quad \text{with } r(x) \in C(\overline{\Omega}) \text{ and } r(x) > \frac{N}{m^-}\ \forall\, x \in \overline{\Omega}\,. \tag{4.52}$$

Condition (4.50) which describes the competition between the growth rates involved in equation (4.49), actually, assures a balance between them and, thus, it represents the *key* for the present study. Such a balance is essential since we are working on a non-homogeneous (eigenvalue) problem for which a minimization technique based on the Lagrange Multiplier Theorem can not be applied in order to find (principal) eigenvalues (unlike the case offered by the homogeneous operators). Thus, in the case of nonlinear non-homogeneous eigenvalue problems the classical theory used in the homogeneous case does not work entirely, but some of its ideas can still be useful and some particular results can still be obtained in some aspects while in other aspects entirely new phenomena can occur. To focus on our case, condition (4.50) together with conditions (4.51) and (4.52) imply

$$\lim_{\|u\|_{0,\Phi_1}\to 0} \frac{\displaystyle\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla u|)\,dx + \int_\Omega \frac{V(x)}{m(x)}|u|^{m(x)}\,dx}{\displaystyle\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}\,dx + \int_\Omega \frac{0,\Phi_1}{q_2(x)}|u|^{q_2(x)}\,dx} = \infty$$

and

$$\lim_{\|u\|_{0,\Phi_1}\to \infty} \frac{\displaystyle\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla u|)\,dx + \int_\Omega \frac{V(x)}{m(x)}|u|^{m(x)}\,dx}{\displaystyle\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}\,dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}\,dx} = \infty\,.$$

In other words, the absence of homogeneity is balanced by the behavior (actually, the blow-up) of the Rayleigh quotient associated to problem (4.49) in the origin and at infinity. The consequences of the above remarks is that the infimum of the Rayleigh quotient associated to problem (4.49) is a real number, i.e.

$$\inf_{u\in W_0^1 L_{\Phi_1}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \Phi_1(|\nabla u|)\,dx + \int_\Omega \Phi_2(|\nabla u|)\,dx + \int_\Omega \frac{V(x)}{m(x)}|u|^{m(x)}\,dx}{\displaystyle\int_\Omega \frac{1}{q_1(x)}|u|^{q_1(x)}\,dx + \int_\Omega \frac{1}{q_2(x)}|u|^{q_2(x)}\,dx} \in \mathbf{R}\,, \tag{4.53}$$

and it will be attained for a function $u_0 \in W_0^1 L_{\Phi_1}(\Omega) \setminus \{0\}$. Moreover, the value in (4.53) represents an eigenvalue of problem (4.49) with the corresponding eigenfunction u_0. However, at this stage we can not say if the eigenvalue described above is the lowest eigenvalue of problem (4.49) or not, even if we are able to show that any λ small enough is not an eigenvalue of (4.49). For the moment this rests an open question. On the other hand, we can prove that any λ superior to the value given by relation (4.53) is also an eigenvalue of problem (4.49). Thus, we conclude that problem (4.49) possesses a continuous family of eigenvalues.

Related with the above ideas we will also discuss the *optimization* of the eigenvalues described by relation (4.53) with respect to potential V, provided that V belongs to a bounded, closed and convex subset of $L^{r(\cdot)}(\Omega)$ (where $r(x)$ is given by relation (4.52)). By optimization we understand the existence of some potentials $V_\star$ and $V^\star$ such that the eigenvalue described in relation (4.53) is minimal or maximal with respect to the set where V lies.

By relation (4.50) it follows that $W_0^1L_{\Phi_1}(\Omega)$ is continuously embedded in $W_0^1L_{\Phi_2}(\Omega)$ (see, e.g. [58, Lemma 2]). Thus, problem (4.49) will be analyzed in the space $W_0^1L_{\Phi_1}(\Omega)$.

We say that $\lambda\in\mathbf{R}$ is an *eigenvalue* of problem (4.49) if there exists $u\in W_0^1L_{\Phi_1}(\Omega)\setminus\{0\}$ such that

$$\begin{aligned}\int_\Omega(a_1(|\nabla u|)+a_2(|\nabla u|))\nabla u\nabla v\,dx \quad &+ \quad \int_\Omega V(x)|u|^{m(x)-2}uv\,dx\\ &- \quad \lambda\int_\Omega(|u|^{q_1(x)-2}+|u|^{q_2(x)-2})uv\,dx=0\,,\end{aligned}$$

for all $v\in W_0^1L_{\Phi_1}(\Omega)$. We point out that if λ is an eigenvalue of problem (4.49) then the corresponding *eigenfunction* $u\in W_0^1L_{\Phi_1}(\Omega)\setminus\{0\}$ is a *weak solution* of problem (4.49).

For each potential $V\in L^{r(\cdot)}(\Omega)$ we define

$$A(V):=\inf_{u\in W_0^1L_{\Phi_1}(\Omega)\setminus\{0\}}\frac{\displaystyle\int_\Omega[\Phi_1(|\nabla u|)+\Phi_2(|\nabla u|)]\,dx+\int_\Omega\frac{V(x)}{m(x)}|u|^{m(x)}\,dx}{\displaystyle\int_\Omega\frac{1}{q_1(x)}|u|^{q_1(x)}\,dx+\int_\Omega\frac{1}{q_2(x)}|u|^{q_2(x)}\,dx}$$

and

$$B(V):=\inf_{u\in W_0^1L_{\Phi_1}(\Omega)\setminus\{0\}}\frac{\displaystyle\int_\Omega[a_1(|\nabla u|)+a_2(|\nabla u|)]|\nabla u|^2\,dx+\int_\Omega V(x)|u|^{m(x)}\,dx}{\displaystyle\int_\Omega|u|^{q_1(x)}\,dx+\int_\Omega|u|^{q_2(x)}\,dx}\,.$$

Thus, we can define two functions $A,\ B:L^{r(\cdot)}(\Omega)\to\mathbf{R}$.

The first result of this section is given by the following theorem.

Theorem 4.5. *Assume that conditions (4.50), (4.51) and (4.52) are fulfilled. Then $A(V)$ is an eigenvalue of problem (4.49). Moreover, there exists $u_V\in W_0^1L_{\Phi_1}(\Omega)\setminus\{0\}$ an eigenfunction corresponding to eigenvalue $A(V)$ such that*

$$A(V)=\frac{\displaystyle\int_\Omega[\Phi_1(|\nabla u_V|)+\Phi_2(|\nabla u_V|)]\,dx+\int_\Omega\frac{V(x)}{m(x)}|u_V|^{m(x)}\,dx}{\displaystyle\int_\Omega\frac{1}{q_1(x)}|u_V|^{q_1(x)}\,dx+\int_\Omega\frac{1}{q_2(x)}|u_V|^{q_2(x)}\,dx}\,.$$

Furthermore, $B(V)\le A(V)$, each $\lambda\in(A(V),\infty)$ is an eigenvalue of problem (4.49), while each $\lambda\in(-\infty,B(V))$ is not an eigenvalue of problem (4.49).

The next result asserts that on each convex, bounded and closed subset of $L^{r(\cdot)}(\Omega)$ function A defined above is bounded from below and attains its minimum. The result is the following:

Theorem 4.6. *Assume that conditions (4.50), (4.51) and (4.52) are fulfilled. Assume that S is a convex, bounded and closed subset of $L^{r(\cdot)}(\Omega)$. Then there exists $V_\star \in S$ which minimizes $A(V)$ on S, i.e.*

$$A(V_\star) = \inf_{V \in S} A(V)\,.$$

Finally, we will focus our attention on the particular case when set S from Theorem 4.6 is a ball in $L^{r(\cdot)}(\Omega)$. Thus, we will denote each closed ball centered in the origin of radius R from $L^{r(\cdot)}(\Omega)$ by $\overline{B}_R(0)$, i.e.

$$\overline{B}_R(0) := \{u \in L^{r(\cdot)}(\Omega);\ \ |u|_{r(\cdot)} \le R\}\,.$$

By Theorem 4.6 we can define function $A_\star : [0,\infty) \to \mathbf{R}$ by

$$A_\star(R) = \min_{V \in \overline{B}_R(0)} A(V)\,.$$

Our result on function $A_\star$ is given by the following theorem:

Theorem 4.7. *a) Function $A_\star$ is not constant and decreases monotonically.*
b) Function $A_\star$ is continuous.

On the other hand, we point out that similar results as those of Theorems 4.6 and 4.7 can be obtained if we notice that on each convex, bounded and closed subset of $L^{r(\cdot)}(\Omega)$ function A defined in Theorem 4.5 is also bounded from above and attains its maximum. It is also easy to remark that we can define a function $A^\star : [0,\infty) \to \mathbf{R}$ by

$$A^\star(R) = \max_{V \in \overline{B}_R(0)} A(V)\,,$$

which has similar properties as $A_\star$.

4.4 The anisotropic case

Let $\Omega \subset \mathbf{R}^N$ $(N \ge 3)$ be a bounded domain with smooth boundary $\partial\Omega$. Consider that for each $i \in \{1, ..., N\}$, φ_i are odd, increasing homeomorphisms from $\mathbf{R}$ onto $\mathbf{R}$, λ is a positive real and $q : \overline{\Omega} \to (1,\infty)$ is a continuous function. The goal of this section is to present some results on the following anisotropic eigenvalue problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_i\left(\varphi_i(\partial_i u)\right) = \lambda |u|^{q(x)-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{4.54}$$

Define

$$\Phi_i(t) = \int_0^t \varphi_i(s)\, ds, \qquad \text{for all } t \in \mathbf{R},\ i \in \{1, ..., N\}\,.$$

Define

$$(p_i)_0 := \inf_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)} \ \text{ and } \ (p_i)^0 := \sup_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)},\ i \in \{1, ..., N\}\,.$$

Assume that conditions (1.10) and (1.12) are satisfied for each $i \in \{1, ..., N\}$. Define the anisotropic Orlicz-Sobolev space $W_0^1 L_{\vec{\Phi}}(\Omega)$ as in Chapter 1. Consider all the definitions, notations and properties of Orlicz-Sobolev and anisotropic Orlicz-Sobolev spaces from Chapter 1.

In the following, for each $i \in \{1, ..., N\}$ we define $a_i : [0, \infty) \to \mathbf{R}$ by,

$$a_i(t) = \begin{cases} \frac{\varphi_i(t)}{t}, & \text{for } \quad t > 0 \\ 0, & \text{for } \quad t = 0\,. \end{cases}$$

Since φ_i are odd we deduce that actually, $\varphi_i(t) = a_i(|t|)t$ for each $t \in \mathbf{R}$ and each $i \in \{1, ..., N\}$.

We say that $\lambda \in \mathbf{R}$ is an *eigenvalue* of problem (4.54) if there exists $u \in W_0^1 L_{\vec{\Phi}}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \left\{ \sum_{i=1}^N |a_i(|\partial_i u|)|\, \partial_i u \partial_i w - \lambda |u|^{q(x)-2} u w \right\} dx = 0$$

for all $w \in W_0^1 L_{\vec{\Phi}}(\Omega)$. For $\lambda \in \mathbf{R}$ an eigenvalue of problem (4.54), function u from the above definition will be called a *weak solution* of problem (4.54) corresponding to eigenvalue λ.

The main results of this section are given by the following theorems:

Theorem 4.8. *Assume that function $q \in C(\overline{\Omega})$ verifies hypothesis*

$$(P^0)_+ < q^- \leq q^+ < (P_0)^\star\,. \tag{4.55}$$

Then any $\lambda > 0$ is an eigenvalue of problem (4.54).

Theorem 4.9. *Assume that function $q \in C(\overline{\Omega})$ satisfies conditions*

$$1 < q^- < (P_0)_- \quad \text{and} \quad q^+ < P_{0,\infty}\,. \tag{4.56}$$

Then there exists $\lambda_\star > 0$ such that any $\lambda \in (0, \lambda_\star)$ is an eigenvalue of problem (4.54).

Theorem 4.10. *Assume that function $q \in C(\overline{\Omega})$ satisfies inequalities*

$$1 < q^- \leq q^+ < (P_0)_-\,. \tag{4.57}$$

Then there exist two positive constants $\lambda_\star > 0$ and $\lambda^\star > 0$ such that any $\lambda \in (0, \lambda_\star) \cup (\lambda^\star, \infty)$ is an eigenvalue of problem (4.54).

Remark 4.4. *By Theorem 4.10 it is not clear if $\lambda_\star < \lambda^\star$ or $\lambda_\star \geq \lambda^\star$. In the first case an interesting question concerns the existence of eigenvalues of problem (4.54) in the interval $[\lambda_\star, \lambda^\star]$.*

In order to state the next result we define

$$\lambda_1 = \inf_{u \in W_0^1 L_{\overrightarrow{\Phi}}(\Omega))\setminus\{0\}} \frac{\displaystyle\int_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u|)\,dx}{\displaystyle\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx}\,,$$

and

$$\lambda_0 = \inf_{u \in W_0^1 L_{\overrightarrow{\Phi}}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega \sum_{i=1}^N a_i(|\partial_i u|)|\partial_i u|^2\,dx}{\displaystyle\int_\Omega |u|^{q(x)}\,dx}\,.$$

Theorem 4.11. *Assume that there exist j_1, j_2, $k \in \{1, ..., N\}$ such that*

$$(p_{j_1})_0 = q^- \quad \text{and} \quad (p_{j_2})^0 = q^+\,, \tag{4.58}$$

and

$$q^+ < \min\{(p_k)_0, (P_0)^\star\}\,. \tag{4.59}$$

Then $0 < \lambda_0 \le \lambda_1$ and every $\lambda \in (\lambda_1, \infty)$ is an eigenvalue of problem (4.54), while no $\lambda \in (0, \lambda_0)$ can be an eigenvalue of problem (4.54).

Remark 4.5. *At this stage we are not able to say whether $\lambda_0 = \lambda_1$ or $\lambda_0 < \lambda_1$. In the latter case an interesting question concerns the existence of eigenvalues of problem (4.54) in the interval $[\lambda_0, \lambda_1]$.*

Chapter 5

Dirichlet eigenvalue problems for difference equations

5.1 Spectral estimates for a nonhomogeneous difference problem

5.1.1 Introduction and main results

Discrete boundary value problems have been intensively studied in the last decade. The modeling of certain nonlinear problems from biological neural networks, economics, optimal control and other areas of study have led to the rapid development of the theory of difference equations (see the monographs of R. P. Agarwal [5] and W. G. Kelley & A. C. Peterson [42] and the papers of R. P. Agarwal, K. Perera & D. O'Regan [6, 7], A. Cabada, A. Iannizzotto & S. Tersian [15], X. Cai & J. Yu [16], J. Yu & Z. Guo [79], G. Zhang & S. Liu [80] and the reference therein).

In view of developing a viable theory of discrete boundary problems, special attention has been given to the study of the *spectrum* of certain *eigenvalue problems*. A classical result in the theory of eigenvalue problems involving difference equations asserts that the spectrum of problem

$$\begin{cases} -\Delta(\Delta u(k-1)) = \lambda u(k), \quad k \in [1,T], \\ u(0) = u(T+1) = 0\,, \end{cases} \tag{5.1}$$

where $T \geq 2$ is an integer, $[1,T]$ is the discrete interval $\{1,2,...,T\}$ and $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator, is *finite* and all the eigenvalues are positive.

On the other hand, some recent advances obtained in [6, 15] show that for some eigenvalue problems involving difference operators the spectrum contains a *continuous family* of eigenvalues.

The goal of this section is to continue the work of the papers mentioned above by presenting a new phenomenon concerning the behaviour of eigenvalues of a nonhomogeneous difference equation. Using the above notations, we are concerned in this section with the eigenvalue problem

$$\begin{cases} -\Delta(\Delta u(k-1)) + |u(k)|^{q-2}u(k) = \lambda g(k)|u(k)|^{r-2}u(k), \quad k \in [1,T], \\ u(0) = u(T+1) = 0\,, \end{cases} \tag{5.2}$$

where q and r are two real numbers satisfying $2 < r < q$ and $g : [1, T] \to (0, \infty)$ is a given function.

We shall prove the existence of *two* positive numbers λ_0 and λ_1, with $\lambda_0 \le \lambda_1$ such that for $\lambda \in (0, \lambda_0)$ problem (5.2) has no non-zero solutions while for any $\lambda \in [\lambda_1, \infty)$ problem (5.2) has non-zero solutions in a specific function space. Moreover, useful estimates will be also given for λ_0 and λ_1 with respect to the initial data q, r, T and g.

In order to describe our result in its full generality we first define the function space

$$H = \{u : [0, T+1] \to \mathbf{R};\ u(0) = u(T+1) = 0\}.$$

Clearly, H is a T-dimensional Hilbert space (see [6]) with the inner product

$$(u, v) = \sum_{k=1}^{T+1} \Delta u(k-1)\Delta v(k-1), \quad \forall\, u, v \in H\,.$$

The associated norm is defined by

$$\|u\| = \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2\right)^{1/2}.$$

We say that $\lambda \in \mathbf{R}$ is an *eigenvalue* of problem (5.2) if there exists $u \in H \setminus \{0\}$ such that

$$\sum_{k=1}^{T+1} \Delta u(k-1)\Delta v(k-1) + \sum_{k=1}^{T} |u(k)|^{q-2}u(k)v(k) - \lambda \sum_{k=1}^{T} g(k)|u(k)|^{r-2}u(k)v(k) = 0, \quad \forall v \in H.$$

Function u in the above definition will be called an *eigenvector* of problem (5.2). The set of all eigenvalues of problem (5.2) will be called the *spectrum* of problem (5.2).

The following theorem represents the main result of this section.

Theorem 5.1. *Let $2 < r < q$, $T \ge 2$ and $g : [1, T] \to (0, \infty)$ be a given function. Then there exist two positive constants λ_0 and λ_1 with $\lambda_0 \le \lambda_1$ such that no $\lambda \in (0, \lambda_0)$ is an eigenvalue of problem (5.2) while any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue of problem (5.2). Moreover, we have*

$$\lambda_1 \le \frac{r}{2}\lambda_0 \quad \text{and} \quad \frac{4}{(T+1)^2|g|_\infty} \le \lambda_0 \le \lambda_1 \le \frac{r(q-2)}{(q-r)\sum_{k=1}^{T} g(k)} \left(\frac{T(q-r)}{q(r-2)}\right)^{\frac{r-2}{q-2}}, \tag{5.3}$$

where $|g|_\infty = \max_{k \in [1,T]} g(k)$.

Notation. For any a and b integers satisfying $a < b$ we denote by $[a, b]$ the discrete interval $\{a, a+1, ..., b\}$.

5.1.2 Some estimates of eigenvalues

In this section we will point out certain remarks on how we can estimate the positive eigenvalues corresponding to positive eigenvectors for problem

$$\begin{cases} -\Delta(\Delta u(k-1)) = \lambda u(k), & k \in [1, T], \\ u(0) = u(T+1) = 0\,. \end{cases} \tag{5.4}$$

In this section, the main result is given by Theorem 5.2 which is of interest in its own right as well:

Theorem 5.2. *Let $\lambda > 0$ be an eigenvalue of problem (5.4) with the property that the corresponding eigenvector $u = \{u(k) : k \in [0, T+1]\}$ is positive, i.e. $u(k) > 0$ for any $k \in [1, T]$. Then we have the estimates*

$$\frac{4}{(T+1)^2} < \lambda \leq \min\left\{1, \frac{1}{T} \cdot \left(1 + \frac{\max\{u(1), u(T)\}}{\min\{u(1), u(T)\}}\right)\right\}. \tag{5.5}$$

Proof. First, we point out certain general remarks on the behavior of $\Delta u(k)$ for $k \in [0, T]$. Since $u(k) > 0$ for $k \in [1, T]$ satisfies equation (5.2) and $\lambda > 0$ we have

$$\Delta(\Delta u(k-1)) = -\lambda u(k) < 0, \quad \forall\, k \in [1, T].$$

Thus, we deduce that sequence $(\Delta u(k))$ is decreasing for $k \in [0, T]$.

Second, we show that the left inequality holds true. In order to do that we start by defining

$$m = \max\{s \in [1, T];\ \ \Delta u(s-1) \geq 0,\ \Delta u(s) < 0\}.$$

Undoubtedly, m can be defined as above since we have $u(T+1) = 0$ and $\Delta u(T) = u(T+1) - u(T) = -u(T) < 0$. (Actually, m is the largest local maximum point of u in $[1, T]$.)

On the other hand, since $\Delta u(m) < 0$ and $(\Delta u(k))$ is a decreasing sequence for $k \in [0, T]$ we notice that

$$\Delta u(k) < 0, \quad \forall\, k \in [m, T],$$

and, thus,

$$u(k+1) < u(k), \quad \forall\, k \in [m, T],$$

i.e. sequence $(u(k))$ is strictly decreasing for $k \in [m, T]$. A similar argument, based on the fact that $\Delta u(m-1) \geq 0$ implies that $\Delta u(k) \geq 0$ for any $k \in [0, m-1]$, i.e. sequence $(u(k))$ is nondecreasing for $k \in [0, m]$.

Adding identities $u(k) - u(k-1) = \Delta u(k-1)$ for $k \in [m+1, T+1]$ we obtain $u(T+1) - u(m) = \sum_{k=m+1}^{T+1} \Delta u(k-1) \geq (T+1-m)\Delta u(T)$, i.e.

$$\frac{-u(m)}{T+1-m} \geq \Delta u(T). \tag{5.6}$$

Since by equation (5.4) we have that

$$\Delta(\Delta u(k-1)) = -\lambda u(k), \quad \forall\, k \in [m, T],$$

summing the above relations with respect to $k \in [m, T]$ we obtain

$$\Delta u(T) - \Delta u(m-1) = -\lambda \sum_{i=m}^{T} u(i).$$

Taking into account that $\Delta u(m-1) \geq 0$ the above equality implies

$$\Delta u(T) \geq -\lambda \sum_{i=m}^{T} u(i).$$

The above inequality, relation (5.6) and the fact that the sequence $(u(k))$ is decreasing for $k \in [m, T]$ yield

$$\frac{-u(m)}{T+1-m} \geq -\lambda \sum_{i=m}^{T} u(i) \geq -\lambda u(m) \sum_{i=m}^{T} 1$$

or

$$\lambda \sum_{i=m}^{T} 1 \geq \frac{1}{T+1-m}. \tag{5.7}$$

In order to go further, we add identities $u(k) - u(k-1) = \Delta u(k-1)$ for $k \in [1, m]$, obtaining that $u(m) - u(0) = \sum_{k=1}^{m} \Delta u(k-1)$. This inequality and the fact that sequence $(\Delta u(k))$ is decreasing for $k \in [0, T]$ imply

$$\Delta u(0) \geq \frac{u(m)}{m}\,. \tag{5.8}$$

Since by equation (5.4) we have that

$$\Delta(\Delta u(k-1)) = -\lambda u(k), \quad \forall\, k \in [1, m]\,,$$

summing the above relations with respect to $k \in [1, m]$ we obtain

$$\Delta u(m) - \Delta u(0) = -\lambda \sum_{i=1}^{m} u(i)\,.$$

But $\Delta u(m) < 0$ and taking into account that relation (5.8) holds true, we infer by the above equality

$$\frac{u(m)}{m} < \lambda \sum_{i=1}^{m} u(i)\,.$$

Using the fact that $(u(k))$ is nondecreasing for $k \in [0, m]$ we find

$$\frac{1}{m} < \lambda \sum_{i=1}^{m} 1 \tag{5.9}$$

Now, by (5.7) and (5.9) we get

$$\lambda \sum_{i=m}^{T} 1 + \lambda \sum_{i=1}^{m} 1 > \frac{1}{T+1-m} + \frac{1}{m}\,.$$

Thus, we conclude that

$$\lambda(T+1) > \frac{4}{T+1}\,,$$

or

$$\lambda > \frac{4}{(T+1)^2}\,.$$

Finally, we prove the second inequality. By equation (5.4) we have

$$\Delta u(k) - \Delta u(k-1) = -\lambda u(k), \quad \forall\, k \in [1, T]\,.$$

Summing the above relations we find

$$u(T) + u(1) = \lambda \sum_{i=1}^{T} u(i)\,.$$

Since $u(k) > 0$ by using the above relation, we find, on the one hand, that

$$u(T) + u(1) \geq \lambda T \min_{k\in[1,T]} u(k)\,,$$

or

$$\frac{u(T) + u(1)}{T \min_{k\in[1,T]} u(k)} \geq \lambda\,, \tag{5.10}$$

and, on the other hand,

$$u(T) + u(1) \geq \lambda(u(1) + u(T))\,,$$

or

$$1 \geq \lambda\,.$$

Furthermore, we notice that if $u(k_0) = \min_{k\in[1,T]} u(k)$ then $k_0 \in \{1, T\}$. Indeed, let us assume by contradiction that $k_0 \in [1, T] \setminus \{1, T\}$. Then, since

$$\Delta u(k_0) - \Delta u(k_0 - 1) = -\lambda u(k_0)\,,$$

or

$$0 \leq u(k_0 + 1) - 2u(k_0) + u(k_0 - 1) = -\lambda u(k_0) < 0\,,$$

we obtain a contradiction. Consequently, $k_0 \in \{1, T\}$. That fact and relation (5.10) yield

$$\frac{1}{T} \cdot \left(1 + \frac{\max\{u(1), u(T)\}}{\min\{u(1), u(T)\}}\right) \geq \lambda\,.$$

Theorem 5.2 is completely proved.

Remark 1. We emphasize that for the estimate in the left-hand side of (5.5) we can give an alternative proof. This idea is described in what follows. The eigenvalues of problem (5.4) can be calculated directly, solving the linear second-order difference equation

$$\Delta(\Delta u(k-1)) + \lambda u(k) = 0\,,$$

(see, e.g. [42, Chapter 3], [11, pp.38]). The eigenvalues of (5.4) are

$$\lambda_k = 2\left(1 - \cos\left(\frac{k\pi}{T+1}\right)\right) = 4\sin^2\left(\frac{k\pi}{2(T+1)}\right), \quad k \in [1, T]\,,$$

and the corresponding eigenvectors are

$$\varphi_k = \left\{0, \sin\left(\frac{k\pi}{T+1}\right), \sin\left(\frac{2k\pi}{T+1},\right), ..., \sin\left(\frac{Tk\pi}{T+1},\right), 0\right\}\,.$$

Note that $0 < \lambda_k < 4$ and the estimate from the left-hand side in (5.5) implies

$$m(T) := \frac{4}{(T+1)^2} < \lambda_1 = 4\sin^2\left(\frac{\pi}{2(T+1)}\right),$$

or, equivalently

$$\frac{1}{(T+1)} < \sin\left(\frac{\pi}{2(T+1)}\right).$$

That fact also follows from the elementary inequality

$$x < \sin\left(\frac{\pi x}{2}\right), \quad \forall\, x \in (0,1).$$

The last inequality is equivalent with the following fact

$$\frac{2}{\pi}x < \sin(x), \quad \forall\, x \in \left(0, \frac{\pi}{2}\right),$$

which geometrically means that the graph of $\sin(x)$ is above the chord which joints the points $(0,0)$ and $(\pi/2, 1)$.

Remark 2. We point out that for a problem of type (5.4) there always exists at least a positive eigenvalue with a positive corresponding eigenfunction, namely, the first eigenvalue (see, e.g., [7] or [3]). Thus, denoting by $\lambda_1([0, T+1])$ the first eigenvalue of equation (5.4), by using Theorem 5.2, we deduce that

$$\frac{4}{(T+1)^2} < \lambda_1([0, T+1]) \le 1. \tag{5.11}$$

Moreover, we point out that the left-hand side inequality in (5.11) is a discrete variant of the celebrated Faber-Krahn inequality which is valid in the continuous case (see, e.g., [26, 44, 46]), since in the particular case when $T = 2$ a simple computation shows that $\lambda_1([0,3]) = 1$ (actually, in this case 1 is the only eigenvalue of the problem), and, thus, the left-hand side of inequality (5.11) can be rewritten in the following way

$$\frac{4}{(T+1)^2}\lambda_1([0,3]) < \lambda_1([0, T+1]), \quad \forall\, T \ge 2.$$

Remark 3. We notice that by a simple computation it can be proved that in the degenerate case $T = 1$ the only eigenvalue of problem (5.4) is $\lambda_1([0,2]) = 2$ while in the case $T = 2$ the two eigenvalues of problem (5.4) are equal to 1. Thus, under these conditions, we have the equality case in the right-hand side of inequality (5.5). In other words, the case when there is equality can occur.

We point out that with a similar proof the result of Theorem 5.2 can be extended to the following:

Theorem 5.3. *Let $p > 1$ be a fixed real number and let $a \ge 1$ and $b \ge a + 2$ be two integers. Consider the problem*

$$\begin{cases} -\Delta(|\Delta u(k-1)|^{p-2}\Delta u(k-1)) = \lambda |u(k)|^{p-2}u(k), \quad k \in [a, b-1], \\ u(a-1) = u(b) = 0. \end{cases} \tag{5.12}$$

Let $\lambda > 0$ *be an eigenvalue of problem (5.12) with the property that the corresponding eigenvector* $u, u(k) > 0$ *for any* $k \in [a, b-1]$*. Then we have the estimates*

$$\frac{2^p}{(b-a+1)^p} < \lambda \leq \min\left\{1, \frac{1}{b-a}\cdot\left(1+\frac{\max\{u(1)^{p-1}, u(T)^{p-1}\}}{\min\{u(1)^{p-1}, u(T)^{p-1}\}}\right)\right\}. \tag{5.13}$$

In the case when $p = 2$, $a = 1$ and $b = T + 1$ in Theorem 5.3 we obtain Theorem 5.2.

Finally, we recall that following the hypotheses of Theorem 5.3 the first eigenvalue, $\lambda_{1,p}([a-1, b])$, is defined from a variational point of view by the so-called Rayleigh quotient, that is

$$\lambda_{1,p}([a-1,b]) = \inf_{u\not\equiv 0} \frac{\sum_{k=a}^{b} |\Delta u(k-1)|^p}{\sum_{k=a}^{b-1} |u(k)|^p}. \tag{5.14}$$

We note that in the case $p = 2$ we will use notation $\lambda_1([a-1, b])$ instead of $\lambda_{1,2}([a-1, b])$. Theorem 5.3 shows that relation (5.11) can be extended thanks to the following relation

$$\frac{4}{(b-a+1)^2} < \lambda_1([a-1,b]) \leq 1. \tag{5.15}$$

5.1.3 Proof of Theorem 5.1

- First, we show the existence of $\lambda_0 > 0$ such that any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem (5.2).

Define the Rayleigh type quotient

$$\lambda_0 = \inf_{u\in H\setminus\{0\}} \frac{\sum_{k=1}^{T+1} |\Delta u(k-1)|^2 + \sum_{k=1}^{T} |u(k)|^q}{\sum_{k=1}^{T} g(k)|u(k)|^r}. \tag{5.16}$$

In a first instance we prove that $\lambda_0 > 0$. In order to show that, we start by pointing out that relations (5.14) and (5.11) imply

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^2 \geq \lambda_1([0, T+1]) \sum_{k=1}^{T} |u(k)|^2 \geq \frac{4}{(T+1)^2} \sum_{k=1}^{T} |u(k)|^2, \quad \forall\, u \in H. \tag{5.17}$$

Since we have $2 < r < q$ we deduce

$$|u(k)|^2 + |u(k)|^q \geq |u(k)|^r, \quad \forall\, u \in H,\ \forall\, k \in [1, T].$$

Summing the above inequalities we obtain

$$\sum_{k=1}^{T} |u(k)|^2 + \sum_{k=1}^{T} |u(k)|^q \geq \sum_{k=1}^{T} |u(k)|^r \geq \frac{1}{|g|_\infty} \sum_{k=1}^{T} g(k)|u(k)|^r, \quad \forall\, u \in H. \tag{5.18}$$

Combining relations (5.17) and (5.18) we infer

$$\begin{aligned}\sum_{k=1}^{T+1}|\Delta u(k-1)|^2+\sum_{k=1}^{T}|u(k)|^q &\geq \min\left\{\frac{4}{(T+1)^2},1\right\}\frac{1}{|g|_\infty}\sum_{k=1}^{T}g(k)|u(k)|^r \\ &= \frac{4}{(T+1)^2|g|_\infty}\sum_{k=1}^{T}g(k)|u(k)|^r, \quad \forall\, u\in H\,.\end{aligned} \tag{5.19}$$

The last inequality shows that

$$\lambda_0 \geq \frac{4}{(T+1)^2|g|_\infty} > 0\,. \tag{5.20}$$

Let us now define, J_1, I_1, J_0, $I_0 : H \to \mathbf{R}$ by

$$J_1(u)=\frac{1}{2}\sum_{k=1}^{T+1}|\Delta u(k-1)|^2+\frac{1}{q}\sum_{k=1}^{T}|u(k)|^q \qquad I_1(u)=\frac{1}{r}\sum_{k=1}^{T}g(k)|u(k)|^r\,,$$

and

$$J_0(u)=\sum_{k=1}^{T+1}|\Delta u(k-1)|^2+\sum_{k=1}^{T}|u(k)|^q \qquad I_0(u)=\sum_{k=1}^{T}g(k)|u(k)|^r\,.$$

Standard arguments imply that J_1, $I_1 \in C^1(H,\mathbf{R})$ with

$$\langle J_1'(u),v\rangle=\sum_{k=1}^{T+1}\Delta u(k-1)\Delta v(k-1)+\sum_{k=1}^{T}|u(k)|^{q-2}u(k)v(k)\,,$$

and

$$\langle I_1'(u),v\rangle=\sum_{k=1}^{T}g(k)|u(k)|^{r-2}u(k)v(k)\,,$$

for any u, $v\in H$.

Lemma 5.1. *Let λ_0 be defined by relation (5.16). Then no $\lambda\in(0,\lambda_0)$ is an eigenvalue of problem (5.2).*

Proof. Indeed, assuming by contradiction that there exists $\lambda\in(0,\lambda_0)$ an eigenvalue of problem (5.2), it follows that we can find $w_\lambda\in H\setminus\{0\}$ such that

$$\langle J_1'(w_\lambda),v\rangle=\lambda\langle I_1'(w_\lambda),v\rangle, \quad \forall v\in H.$$

Letting $v=w_\lambda$ we deduce $\langle J_1'(w_\lambda),w_\lambda\rangle=\lambda\langle I_1'(w_\lambda),w_\lambda\rangle$, or

$$J_0(w_\lambda)=\lambda I_0(w_\lambda)\,.$$

Since $w_\lambda\neq 0$ we have that $J_0(w_\lambda)>0$ and, thus, $I_0(w_\lambda)>0$. Combining that fact with the ideas that $\lambda\in(0,\lambda_0)$ and $\lambda_0=\inf_{u\in H\setminus\{0\}}\frac{J_0(u)}{I_0(u)}$ we infer

$$J_0(w_\lambda)\geq\lambda_0 I_0(w_\lambda)>\lambda I_0(w_\lambda)=J_0(w_\lambda),$$

which is a contradiction. The proof of Lemma 5.1 is complete.

• Secondly, we show that there exists λ_1 such that any $\lambda \in (\lambda_1, \infty)$ is an eigenvalue of problem (5.2).

For any $\lambda > 0$ we define functional $S_\lambda : H \to \mathbf{R}$ by

$$S_\lambda(u) = J_1(u) - \lambda I_1(u), \quad \forall\, u \in H\,.$$

We notice that $S_\lambda \in C^1(H, \mathbf{R})$ with the derivative given by

$$\langle S_\lambda^{'}(u), v\rangle = \langle J_1^{'}(u), v\rangle - \lambda\langle I_1^{'}(u), v\rangle, \quad \forall\, u,\ v \in H\,.$$

Thus, λ is an eigenvalue of problem (5.2) if and only if there exists $u_\lambda \in H \setminus \{0\}$ a critical point of S_λ.

Lemma 5.2. *For any $\lambda \in (0, \infty)$ functional S_λ is coercive, i.e.* $\lim_{\|u\|\to\infty} S_\lambda(u) = \infty$.

Proof. It is obvious that

$$S_\lambda(u) \geq \frac{1}{2}\|u\|^2 + \frac{1}{q}\sum_{k=1}^{T}|u(k)|^q - \frac{|g|_\infty}{r}\sum_{k=1}^{T}|u(k)|^r\,,$$

for any $u \in H$. For any $m \geq 2$ let us denote

$$|u|_m = \left(\sum_{k=1}^{T}|u(k)|^m\right)^{1/m}\,.$$

It is not difficult to notice that each $|\cdot|_m$, $m \geq 2$, is a norm on H. Since H is a finite dimensional Hilbert space we deduce that for any m_1, $m_2 \geq 2$ the norms $|\cdot|_{m_1}$, $|\cdot|_{m_2}$ and $\|\cdot\|$ are equivalent.

The above pieces of information imply that there exist two positive constants C_1 and C_2 such that

$$S_\lambda(u) \geq \frac{1}{2}\|u\|^2 + C_1\|u\|^q - C_2\|u\|^r\,,$$

for any $u \in H$. Since $2 < r < q$, the proof of Lemma 5.2 is complete.

Define

$$\lambda_1 = \inf_{u\in H\setminus\{0\}} \frac{\dfrac{1}{2}\displaystyle\sum_{k=1}^{T+1}|\Delta u(k-1)|^2 + \dfrac{1}{q}\sum_{k=1}^{T}|u(k)|^q}{\dfrac{1}{r}\displaystyle\sum_{k=1}^{T}g(k)|u(k)|^r}\,. \tag{5.21}$$

Due to (5.16), a simple estimate shows that

$$r\min\left\{\frac{1}{2}, \frac{1}{q}\right\}\lambda_0 \leq \lambda_1 \leq r\max\left\{\frac{1}{2}, \frac{1}{q}\right\}\lambda_0.$$

Since $2 < r < q$, we clearly have

$$\frac{r}{q}\lambda_0 \leq \lambda_1 \leq \frac{r}{2}\lambda_0. \tag{5.22}$$

In particular, (5.20) and the left hand size of (5.22) imply $\lambda_1 > 0$.

Lemma 5.3. *Any $\lambda \in (\lambda_1, \infty)$ is an eigenvalue of problem (5.2).*

Proof. We fix $\lambda \in (\lambda_1, \infty)$. By Lemma 5.2 we deduce that S_λ is coercive. On the other hand, it is clear that functional S_λ is weakly lower semi-continuous. These two facts enable us to apply Theorem 1.2 in [75] in order to prove that there exists $u_\lambda \in H$ a global minimum point of S_λ.

Next, we show that u_λ is not trivial. Indeed, since $\lambda_1 = \inf_{u \in H,\, u \neq 0} \frac{J_1(u)}{I_1(u)}$ and $\lambda > \lambda_1$ it follows that there exists $v_\lambda \in H$ such that

$$J_1(v_\lambda) < \lambda I_1(v_\lambda)\,,$$

or

$$S_\lambda(v_\lambda) < 0\,.$$

In particular, $\inf_H S_\lambda < 0$, and we conclude that $u_\lambda \neq 0$.

Next, we show that λ is an eigenvalue of problem (5.2). Let $v \in H$ fixed. The above property of u_λ gives that

$$\frac{d}{d\epsilon} S_\lambda(u_\lambda + \epsilon v)|_{\epsilon=0} = 0\,,$$

or

$$\langle J_1'(u_\lambda), v\rangle - \lambda \langle I_1'(u_\lambda), v\rangle = 0, \quad \forall\, v \in H\,,$$

that means λ is an eigenvalue of problem (5.2). The proof of Lemma 5.3 is complete.

• Next, we show that λ_1 is also an eigenvalue of problem (5.2). In order to do that we first prove the following result.

Lemma 5.4. $\lim_{\|u\| \to 0} \frac{J_0(u)}{I_0(u)} = \lim_{\|u\| \to \infty} \frac{J_0(u)}{I_0(u)} = \infty\,.$

Proof. Considering again norms, $|\cdot|_m$, $m \geq 2$, defined in Lemma 5.2 and recalling that they are equivalent with norm $\|\cdot\|$ we find that there exist two positive constants D_1 and D_2 such that

$$\frac{J_0(u)}{I_0(u)} \geq \frac{\|u\|^2 + D_1\|u\|^q}{D_2\|u\|^r}\,, \quad \forall u \in H \setminus \{0\}.$$

Now taking into account that $2 < r < q$, the conclusion of Lemma 5.4 immediately holds.

Lemma 5.5. *The real number λ_1, given by relation (5.21), is an eigenvalue of problem (5.2).*

Proof. Let (λ_n) be a sequence in $\mathbf{R}$ such that $\lambda_n \searrow \lambda_1$ as $n \to \infty$. By Lemma 5.3 we deduce that for each n there exists $u_n \in H \setminus \{0\}$ such that

$$\langle J_1'(u_n), v\rangle - \lambda_n \langle I_1'(u_n), v\rangle = 0, \quad \forall\, v \in H\,. \tag{5.23}$$

Taking $v = u_n$ in the above equality we find

$$J_0(u_n) = \lambda_n I_0(u_n), \quad \forall\, n\,. \tag{5.24}$$

The above equality and Lemma 5.4 imply that (u_n) is a bounded sequence in H. Indeed, assuming by contradiction that (u_n) is not bounded in H it follows that passing eventually to a subsequence, still denoted by (u_n) we have $\|u_n\| \to \infty$. On the other hand, the fact that $\lambda_n \searrow \lambda_1$ and relation (5.24) imply that for each n large enough the following holds true

$$\frac{J_0(u_n)}{I_0(u_n)} = \lambda_n \leq \lambda_1 + 1\,.$$

Lemma 5.4 shows that the above inequality and the fact that $\|u_n\| \to \infty$ lead to a contradiction. Consequently, (u_n) is bounded in H. We deduce the existence of $u \in H$ such that, passing eventually to a subsequence, u_n converges to u in H. Passing to the limit as $n \to \infty$ in (5.23) we get

$$\langle J_1'(u), v\rangle - \lambda_1 \langle I_1'(u), v\rangle = 0, \quad \forall\, v \in H\,,$$

i.e. λ_1 is an eigenvalue of problem (5.2) provided that $u \neq 0$.

Finally, we explain why $u \neq 0$. Assuming by contradiction that $u = 0$ we deduce that u_n converges to 0 in H. By relation (5.24) we deduce that for any n the following equality holds

$$\frac{J_0(u_n)}{I_0(u_n)} = \lambda_n\,.$$

Passing to the limit as $n \to \infty$ and taking into account the result of Lemma 5.4 and the fact that $\lambda_n \searrow \lambda_1$ we obtain a contradiction. The proof of Lemma 5.5 is complete.

• Finally, we point out that the conclusion of Theorem 5.1 holds true.

Proof of Theorem 5.1. In order to obtain the first part, it is enough to combine Lemmas 5.1, 5.3 and 5.5; in particular, we clearly have $\lambda_0 \leq \lambda_1$. The first two inequalities of (5.3) come from (5.22) and (5.20), respectively.

It remains to prove the right hand side inequality of (5.3), i.e., $\lambda_1 \leq A$, where we use notation $A = \frac{r(q-2)}{(q-r)\sum_{k=1}^{T} g(k)} \left(\frac{T(q-r)}{q(r-2)}\right)^{\frac{r-2}{q-2}}$. Fix $\tilde{u} \in H \setminus \{0\}$ by $\tilde{u}(k) = s > 0$, $k \in [1, T]$. Due to (5.21), we have

$$\lambda_1 \leq \frac{\frac{1}{2}\sum_{k=1}^{T+1} |\Delta \tilde{u}(k-1)|^2 + \frac{1}{q}\sum_{k=1}^{T} |\tilde{u}(k)|^q}{\frac{1}{r}\sum_{k=1}^{T} g(k)|\tilde{u}(k)|^r} = \frac{r(s^2 + \frac{T}{q}s^q)}{s^r \sum_{k=1}^{T} g(k)}.$$

Taking function $h : (0, \infty) \to (0, \infty)$ defined by

$$h(s) = \frac{r(s^2 + \frac{T}{q}s^q)}{s^r \sum_{k=1}^{T} g(k)},$$

one can easily show that its minimum is attained at the point $s_0 = \left(\frac{q(r-2)}{T(q-r)}\right)^{\frac{1}{q-2}}$, the minimum value being $h(s_0) = A$. This concludes the proof.

Remark 4. We notice that the spectrum of problem (5.2) is not completely described by our study. Although we have estimates for λ_0 and λ_1, at this stage we are not able to say if $\lambda_0 = \lambda_1$ or $\lambda_0 < \lambda_1$. Note that λ_0 and λ_1 are very close to each other whenever r is close to 2; in that sense, see the first inequality in (5.3). Due to the nonhomogeneous nature of problem (5.2), we are strongly convinced that we usually have $\lambda_0 < \lambda_1$, i.e., there is a gap between λ_0 and λ_1. If so, the problem of the existence/nonexistence of eigenvalues in the interval $[\lambda_0, \lambda_1)$ should be elucidated.

5.2 Eigenvalue problems for anisotropic discrete boundary value problems

In this section we present some results regarding the existence of solutions for the discrete boundary value problem

$$\begin{cases} -\Delta(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)) = \lambda |u(k)|^{q(k)-2}u(k), \quad k \in \mathbf{Z}[1,T], \\ u(0) = u(T+1) = 0\,, \end{cases} \tag{5.25}$$

where $T \geq 2$ is a positive integer and $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator. Here and hereafter, we denote by $\mathbf{Z}[a,b]$ the discrete interval $\{a, a+1, ..., b\}$ where a and b are integers and $a < b$. Moreover, we assume that functions $p : \mathbf{Z}[0,T] \to [2,\infty)$ and $q : \mathbf{Z}[1,T] \to [2,\infty)$ are bounded while λ is a positive constant.

We note that problem (5.25) is the discrete variant of the variable exponent anisotropic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) = \lambda |u|^{q(x)-2}u, & \text{for} \quad x \in \Omega \\ u = 0, & \text{for} \quad x \in \partial\Omega\,, \end{cases} \tag{5.26}$$

where $\Omega \subset \mathbf{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary, $\lambda > 0$ is a real number, and $p_i(x)$, $q(x)$ are continuous on $\overline{\Omega}$ such that $N > p_i(x) \geq 2$ and $q(x) > 1$ for any $x \in \overline{\Omega}$ and all $i \in \mathbf{Z}[1,N]$. Problem (5.26) was recently analyzed by M. Mihăilescu, P. Pucci & V. Rădulescu in [53, 54].

Using critical point theory we can establish the existence of a continuous spectrum of eigenvalues for problems of type (5.25). The idea is to transfer the problem of the existence of solutions for problem (5.25) into the problem of existence of critical points for some associated energy functional. On the other hand, we point out that, to our best knowledge, discrete problems like (5.25), involving anisotropic exponents, have not yet been discussed. Thus, the present study can be regarded as a contribution in this direction.

We are interested in finding weak solutions for problems of type (5.25). For this purpose we define the function space

$$H = \{u : \mathbf{Z}[0, T+1] \to \mathbf{R};\ \text{such that } u(0) = u(T+1) = 0\}.$$

Clearly, H is a T-dimensional Hilbert space (see [6]) with the inner product

$$(u,v) = \sum_{k=1}^{T+1} \Delta u(k-1)\Delta v(k-1), \quad \forall\, u,v \in H\,.$$

This associated norm is defined by

$$\|u\| = \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2\right)^{1/2}.$$

By a *weak solution* for problem (5.25) we understand a function $u \in H$ such that

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta v(k-1) - \lambda \sum_{k=1}^{T} |u(k)|^{q(k)-2}u(k)v(k) = 0\,,$$

for any $v \in H$.

Denote for short $\max_{k\in\mathbf{Z}[a,b]} p(k)$ by $\max_{\mathbf{Z}[a,b]} p$ and $\min_{k\in\mathbf{Z}[a,b]} p(k)$ by $\min_{\mathbf{Z}[a,b]} p$.

The main results of this section are the following.

Theorem 5.4. *Assume that functions p and q verify hypothesis*

$$\max_{\mathbf{Z}[0,T]} p < \min_{\mathbf{Z}[1,T]} q\,. \tag{5.27}$$

Then for any $\lambda > 0$ problem (5.25) has a nontrivial weak solution.

Theorem 5.5. *Assume that functions p and q verify hypothesis*

$$\max_{\mathbf{Z}[1,T]} q < \min_{\mathbf{Z}[0,T]} p\,. \tag{5.28}$$

Then there exists $\lambda^{\star\star} > 0$ such that for any $\lambda > \lambda^{\star\star}$ problem (5.25) has a nontrivial weak solution.

Theorem 5.6. *Assume that functions p and q verify hypothesis*

$$\min_{\mathbf{Z}[1,T]} q < \min_{\mathbf{Z}[0,T]} p\,. \tag{5.29}$$

Then there exists $\lambda^{\star} > 0$ such that for any $\lambda \in (0, \lambda^{\star})$ problem (5.25) has a nontrivial weak solution.

Remark 1. We point out that if relation (5.29) is verified then relation (5.28) is fulfilled, too. Consequently, the result of Theorem 5.5 can be completed with the conclusion of Theorem 5.6. More exactly, we deduce the following corollary.

Corollary 5.1. *Assume that functions p and q verify hypothesis*

$$\min_{\mathbf{Z}[1,T]} q < \min_{\mathbf{Z}[0,T]} p\,.$$

Then there exist $\lambda^{\star} > 0$ and $\lambda^{\star\star} > 0$ such that for any $\lambda \in (0, \lambda^{\star}) \cup (\lambda^{\star\star}, \infty)$ problem (5.25) possesses a nontrivial weak solution.

Remark 2. On the other hand, we point out that the result of Theorem 5.6 holds true in situations that extend relation (5.28) since in relation (5.29) we could have

$$\min_{\mathbf{Z}[1,T]} q < \min_{\mathbf{Z}[0,T]} p < \max_{\mathbf{Z}[1,T]} q\,.$$

Bibliography

[1] E. Acerbi and G. Mingione, Regularity results for a class of functionals with nonstandard growth, *Arch. Rational Mech. Anal.* **156** (2001), 121-140.

[2] E. Acerbi and G. Mingione, Gradient estimates for the $p(x)$-Laplacean system, *J. Reine Angew. Math.* **584** (2005), 117-148.

[3] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[4] D.R. Adams and L.I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 314, Springer-Verlag, Berlin, 1996.

[5] R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker Inc., 2000.

[6] R. P. Agarwal, K. Perera and D. O'Regan, Multiple positive solutions of singular and nonsingular discrete problems via variational methods, *Nonlinear Analysis* **58** (2004), 69-73.

[7] R. P. Agarwal, K. Perera and D. O'Regan, Multiple positive solutions of singular discrete p–Laplacian problems via variational methods, *Advances in Difference Equations* 2005:2 (2005) 93-99.

[8] A. Anane, Simplicité et isolation de la première valeur propre du p-laplacien avec poids, *C. R. Acad. Sci. Paris, Sér. I Math.* **305** (1987), 725-728.

[9] M. S. Ashbaugh and E. M. Harrell, Maximal and minimal eigenvalues and their associated nonlinear equations, *J. Math. Phys.* **28** (1987), 1770-1786.

[10] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leyden, 1976.

[11] C. Bereanu, *Topological degree methods for some nonlinear problems*, Ph. D. Thesis, Université Catholique de Louvain, 2006.

[12] J. F. Bonder and L. M. Del Pezzo, An optimization problem for the first eigenvalue of the p-Laplacian plus a potential, *Commun. Pure Appl. Anal.* **5** (2006), 675-690.

[13] H. Brezis, *Analyse Fonctionnelle. Théorie, Méthodes et Applications*, Masson, Paris, 1992.

[14] D. Bucur and D. Daners, An alternative approach to the Faber-Krahn inequality for Robin problems, Preprint Ulmer Seminare über Funktionalanalysis und Differentialgleichungen 2009.

[15] A. Cabada, A. Iannizzotto and S. Tersian, Multiple solutions for discrete boundary value problems, *J. Math. Anal. Appl.* **356** (2009), 418-428.

[16] X. Cai and J. Yu, Existence theorems for second-order discrete boundary value problems, *J. Math. Anal. Appl.* **320** (2006), 649-661.

[17] Y. Chen, S. Levine and R. Rao, Variable exponent, linear growth functionals in image processing, *SIAM J. Appl. Math.* **66** (2006), 1383-1406.

[18] Ph. Clément, M. García-Huidobro, R. Manásevich, and K. Schmitt, Mountain pass type solutions for quasilinear elliptic equations, *Calc. Var.* **11** (2000), 33-62.

[19] Ph. Clément, B. de Pagter, G. Sweers, and F. de Thélin, Existence of solutions to a semilinear elliptic system through Orlicz-Sobolev spaces, *Mediterr. J. Math.* **1** (2004), 241-267.

[20] L. Diening, *Theorical and numerical results for electrorheological fluids*, Ph.D. thesis, University of Frieburg, Germany, 2002.

[21] D. E. Edmunds, J. Lang and A. Nekvinda, On $L^{p(x)}$ norms, *Proc. Roy. Soc. London Ser. A* 455 (1999) 219-225.

[22] D. E. Edmunds and J. Rákosník, Density of smooth functions in $W^{k,p(x)}(\Omega)$, *Proc. Roy. Soc. London Ser. A* 437 (1992) 229-236.

[23] D. E. Edmunds and J. Rákosník, Sobolev embedding with variable exponent, *Studia Math.* 143 (2000) 267-293.

[24] H. Egnell, Extremal properties of the first eigenvalue of a class of elliptic eigenvalue problems, *Ann. Sc. Norm. Sup. Pisa* **14** (1987), 1-48.

[25] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47** (1974), 324-353.

[26] C. Faber, Beweis das unter allen homogenen Membranen yon gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt, *Sitzungsber. Bayer. Akad. der Wiss. Math. Phys.*, Munich 1923, pp. 169-172.

[27] X. Fan, Remarks on eigenvalue problems involving the $p(x)$-Laplacian, *J. Math. Anal. Appl.* **352** (2009), 85-98.

[28] X. Fan and X. Han, 'Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbf{R}^N$', *Nonlinear Analysis* 59 (2004) 173-188.

[29] X. L. Fan and Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Anal.* **52** (2003), 1843-1852.

[30] X. Fan, Q. Zhang and D. Zhao, Eigenvalues of $p(x)$-Laplacian Dirichlet problem, *J. Math. Anal. Appl.* **302** (2005), 306-317.

[31] X. L. Fan and D. Zhao, On the Spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263** (2001), 424-446.

[32] I. Fragalà, F. Gazzola, and B. Kawohl, Existence and nonexistence results for anisotropic quasilinear equations, *Ann. Inst. H. Poincaré, Analyse Non Linéaire* **21** (2004), 715–734.

[33] M. Garciá-Huidobro, V.K. Le, R. Manásevich, and K. Schmitt, On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz-Sobolev space setting, *Nonlinear Differential Equations Appl. (NoDEA)* **6** (1999), 207-225.

[34] L. Gasiński and N. S. Papagiorgiu, *Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems*, Chapman & Hall, 2005.

[35] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1998.

[36] T. Giorgi and R. G. Smits, Monotonicity results for the principal eigenvalue of the generalized Robin problem, *Illinois J. Math.* **49** (4) (2005), 1133-1143.

[37] J.P. Gossez, Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients, *Trans. Amer. Math. Soc.* **190** (1974), 163-205.

[38] T. C. Halsey, Electrorheological fluids, *Science* **258** (1992), 761-766.

[39] P. Harjulehto, P. Hästö, Ú. V. Lê and M. Nuortio, *Overview of differential equations with non-standard growth*, *Nonlinear Analysis*, in press.

[40] A. Henrot, *Extremum Problems for Eigenvalues of Elliptic Operators*, Birkhäuser, 2006.

[41] R. Kajikia, A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations, *J. Funct. Anal.* **225** (2005), 352-370.

[42] W. G. Kelley and A. C. Peterson, *Difference Equations. An Introduction with Applications*, Harcourt Academic Press, 2001.

[43] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.* 41 (1991) 592-618.

[44] E. Krahn, Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises, *Math. Ann.* **94** (1925), 97-100.

[45] A. Kristály, M. Mihăilescu, V. Rădulescu and S. Tersian, Spectral estimates for a nonhomogeneous difference problem, *Communications in Contemporary Mathematics*, in press.

[46] E. H. Lieb, On the lowest eigenvalue of the Laplacian for the intersection of two domains, *Invent. Math.* **74** (1983), 441-448.

[47] M. Mihăilescu, An eigenvalue problem possessing a continuous family of eigenvalues plus an isolated eigenvalue, *Communications on Pure and Applied Analysis*, in press.

[48] M. Mihăilescu and G. Moroşanu, Existence and multiplicity of solutions for an anisotropic elliptic problem involving variable exponent growth conditions, *Applicable Analysis* **89** (2) (2010), 257-271.

[49] M. Mihăilescu and G. Moroşanu, On an eigenvalue problem for an anisotropic elliptic equation involving variable exponents, *Glasgow Mathematical Journal* **52** (2010), 517-527.

[50] M. Mihăilescu and G. Moroşanu, Eigenvalues of the Laplace operator with nonlinear boundary conditions, *Taiwanese Journal of Mathematics*, in press.

[51] M. Mihăilescu, G. Moroşanu and V. Rădulescu, Eigenvalue problems in anisotropic Orlicz-Sobolev spaces, *C. R. Acad. Sci. Paris, Ser.*, I **347** (2009), 521-526.

[52] M. Mihăilescu, G. Moroşanu and V. Rădulescu, Eigenvalue problems for anisotropic elliptic equations: an Orlicz-Sobolev space setting, *Nonlinear Analysis* **73** (2010), 3239-3252.

[53] M. Mihăilescu, P. Pucci and V. Rădulescu, Nonhomogeneous boundary value problems in anisotropic Sobolev spaces, *C. R. Acad. Sci. Paris, Ser. I* 345 (2007) 561-566.

[54] M. Mihăilescu, P. Pucci and V. Rădulescu, Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent, *J. Math. Anal. Appl.* **340** (2008), 687-698.

[55] M. Mihăilescu and V. Rădulescu, Continuous spectrum for a class of nonhomogeneous differential operators, *Manuscripta Mathematica* 125 (2008) 157-167.

[56] M. Mihăilescu and V. Rădulescu, Neumann problems associated to nonhomogeneous differential operators in Orlicz-Sobolev spaces, *Annales de l'Institut Fourier* **58** (6) (2008), 2087-2111.

[57] M. Mihăilescu and V. Rădulescu, Spectrum consisting in an unbounded interval for a class of nonhomogeneous differential operators, *Bulletin of the London Mathematical Society* **40** (2008), 972-984.

[58] M. Mihăilescu and V. Rădulescu, Eigenvalue problems associated to nonhomogeneous differential operators in Orlicz-Sobolev spaces, *Analysis and Applications* **6** (2008), No. 1, 1-16.

[59] M. Mihăilescu and V. Rădulescu, A continuous spectrum for nonhomogeneous differential operators in Orlicz-Sobolev spaces, *Mathematica Scandinavica* **104** (2009), 132-146.

[60] M. Mihăilescu and V. Rădulescu, Sublinear eigenvalue problems associated to the Laplace operator revisited, *Israel J. Math.*, in press.

[61] M. Mihăilescu and V. Rădulescu, Concentration phenomena in nonlinear eigenvalue problems with variable exponents and sign-changing potential, *Journal d'Analyse Mathématique*, in press.

[62] M. Mihăilescu, V. Rădulescu and D. Repovš, On a non-homogeneous eigenvalue problem involving a potential: an Orlicz-Sobolev space setting, *J. Math. Pures Appliquées (Journal de Liouville)* **93** (2010), 132-148.

[63] M. Mihăilescu, V. Rădulescu and D. Stancu-Dumitru, On a Caffarelli-Kohn-Nirenberg type inequality in bounded domains involving variable exponent growth conditions and applications to PDE's, *Complex Variables-Elliptic Equations*, in press.

[64] M. Mihăilescu, V. Rădulescu and S. Tersian, Eigenvalue Problems for Anisotropic Discrete Boundary Value Problems, *Journal of Difference Equations and Applications* **15** (2009), 557-567.

[65] M. Mihăilescu and D. Stancu-Dumitru, On an eigenvalue problem involving the $p(x)$-Laplace operator plus a non-local term, *Differential Equations & Applications* **1** (2009), 367-378.

[66] J. Musielak, *Orlicz Spaces and Modular Spaces*, Springer, Berlin, 1983.

[67] S. M. Nikol'skii, On imbedding, continuation and approximation theorems for differentiable functions of several variables, *Russian Math. Surv.* **16** (1961), 55–104.

[68] L. Payne and H. Weinberger, An optimal Poincaré inequality for convex domains, *Arch. Rational Mech. Anal.* **5** (1960), 286-292.

[69] J. Rákosník, Some remarks to anisotropic Sobolev spaces I, *Beiträge zur Analysis* **13** (1979), 55–68.

[70] J. Rákosník, Some remarks to anisotropic Sobolev spaces II, *Beiträge zur Analysis* **15** (1981), 127–140.

[71] M.M. Rao and Z.D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, Inc., New York, 1991.

[72] M. Ruzicka, Flow of shear dependent electrorheological fluids, *C. R. Acad. Sci. Paris Ser. I Math.* **329** (1999), 393-398.

[73] M. Ruzicka, *Electrorheological Fluids Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2002.

[74] S. Samko and B. Vakulov, Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators, *J. Math. Anal. Appl.* 310 (2005) 229-246.

[75] M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, Heidelberg, 1996.

[76] M. Troisi, Teoremi di inclusione per spazi di Sobolev non isotropi, *Ricerche Mat.* **18** (1969), 3–24.

[77] L. Ven'–tuan, On embedding theorems for spaces of functions with partial derivatives of various degree of summability, *Vestnik Leningrad Univ.* **16** (1961), 23–37.

[78] M. Willem, *Analyse Harmonique Réelle*, Hermann, 1995.

[79] J. Yu and Z. Guo, On boundary value problems for a discrete generalized Emden-Fowler equation, *J. Math. Anal. Appl.* **231** (2006), 18-31.

[80] G. Zhang and S. Liu, On a class of semipositone discrete boundary value problem, *J. Math. Anal. Appl.* **325** (2007), 175-182.

[81] V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, *Math. USSR Izv.* **29** (1987), 33-66.