Customer Lock-In With Long-Term Contracts

Zsolt Macskasi
Northwestern University
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Abstract

We consider a horizontally differentiated industry with two firms and two time periods. We allow for customers’ relative preferences towards firms to change over time. Firms are allowed to offer short-term and long-term contracts to customers. The main goal is to investigate the effects of the presence of long-term contracts. On one hand, long-term contracts create social inefficiency, because as their preferences change, customers might be forced to purchase from a firm, which they no longer like the most. The effects on profits and consumer welfare are not as clearcut. Although long-term contracts lock-in consumers, they must be offered at a discount relative to short-term contracts. It depends on the parameter-configuration, which of these effects dominates. Sometimes, the ability of firms to offer long-term contracts is beneficial to consumers. Moreover, the benefits of long-term contracts are not equally distributed among consumers, so some of them might be better off, but at the same time, some of them might be worse off.

1 Introduction

It has a lot of practical relevance whether long-term contracts are anti-competitive or not. It would seem natural to say that long-term contracts provide market power to firms. These contracts lock-in customers into buying the same product today as they did in the past. On the other hand, competition forces firms to compensate consumers for being locked-in. Thus, to study the welfare and distributional effects rigorously, we need a model.

The general topic of this paper is price competition and consumer behavior in a dynamic duopoly model. Usually, the purpose of these models is to examine how prices (profits) and market shares evolve over time. The first issue raises the question whether something could happen over time, which would increase or decrease firms’ market power. For example, the product/service can be such that consumers obtain a lot of specific knowledge while using it, which makes it costly for them to switch to an alternative provider.
Klemperer (1995) provides an excellent taxonomy about the various sources of these switching costs. It was shown in several papers that these consumer switching costs make the market less competitive, as seen for example, in the seminal works of Klemperer (1987a, 1987b, 1987c). This result seems quite obvious in a two-period model, since in the second period firms are able to charge higher prices due to consumer lock-in. More interestingly, as Beggs and Klemperer (1992) show, the result holds in an infinite-horizon world, too. In this case, firms face two opposing incentives in every period: on one hand, to charge low prices in order to build future market share and, on the other hand, to charge high prices in order to rip off locked-in consumers. The article shows that the second incentive is stronger; steady state prices are higher than static prices.¹ In real world, sometimes competition forces service providers to take away the burden of switching costs from consumers. Chen (1997) presents a model, in which suppliers are allowed to pay consumers to switch. Interestingly, the result of softened price competition still holds.

A special form of price discrimination can have similar effects to consumer switching costs. Caminal and Matutes (1990) set up a model, in which firms precommit to a more favorable treatment of loyal customers than of new customers. They consider two cases, precommitting to a specific price, and precommitting to a specific discount that is offered to loyal customers. In a symmetric equilibrium, these policies make patronizing the same firm more attractive than switching. Hence, they can be viewed as implicit switching costs. It is important to notice that unlike consumer switching costs, these costs are endogenous. It turns out that they might also yield a completely different result. In particular, Caminal and Matutes (1990) show that, compared to a standard static model, commitment to prices leads to lower, whereas commitment to discounts leads to higher profits. The commitment in this model is a unilateral action taken by firms; consumers do not have to commit to remain loyal. This assumption is a plausible one when modeling frequent flyer programs or retail store coupons. However, we see many instances (most notably subscription markets), where firms and customers make a mutually binding contract. For example, in a typical cellular phone contract the provider guarantees the service at a set price for a determined length of time. The customer must pay at regular time intervals, until the contract ends. He could breach the contract in the meantime, but then he must pay a penalty. Curiously, we have not seen any theoretical works that have dealt with this particular problem.

There have been, of course, works on the welfare effects of long-term contracts. In Aghion and Bolton (1987), for example, an incumbent monopolist and a buyer make a mutually beneficial contract, which in turn, has the consequence of making entry less likely. They effectively form a coalition against the potential entrant. In a world of incomplete information, this contract could reduce social welfare, because

¹Taylor (1999) argues that softened price competition due to consumer switching costs is a pathological result of duopoly structure, because already with three firms, one gets perfect competition. However, this result is an artifact of considering a homogenous product industry.
while beneficial for the incumbent and the buyer, it has really bad consequences on the entrant.

On the other hand, Fudenberg and Tirole (2000) show that long-term contracts can be welfare-improving if they reduce socially inefficient switching. Switching between providers is socially inefficient if it is due to the fact that firms price discriminate in favor of their competitors’ former clients. This practice, also called poaching, has got theoretical attention only recently. We know four papers on this arena, the aforementioned one from Fudenberg and Tirole, Chen (1997), Taylor (1999) and Villas-Boas (1999). Common in these papers is that firms observe past individual choices, and they charge a more favorable price to their potential new clients than to their old clients. Customers choose first the most preferred firm, but later some of them switch away to the less preferred firm, because of the low poaching price. Hence, switching is socially inefficient. Notice that while consumer switching costs and long-term contracts make switching more difficult, poaching, on the contrary, makes it more attractive. Fudenberg and Tirole (2000) also call this practice behavior-based price discrimination, because the price that a consumer faces in the second period depends on which of the firms did he choose in the first period.\(^2\)

This paper aims to show how prices of short-term and long-term contracts are determined in a competitive environment. The model best could be applied to a subscription market where consumers use some service continuously, and make payments at regular time intervals. We assume that the service is horizontally differentiated. In particular, we consider the circular city model, in which customers are distributed along a circle, and firms are located at the endpoints of a diameter. It is quite obvious that if consumer preferences were fixed, then in equilibrium nobody would switch from one firm to another. Hence, we allow for preferences to change over time. We consider an idiosyncratic, zero-mean taste shock that relocates customers along the circle. We assume that both customers and firms are rational agents who have correct anticipations about future prices and consumer choices. We find three types of symmetric equilibria. In one type of equilibrium, firms sell both short-term and long-term contracts. Firms effectively discriminate customers into four segments. Customers with strong preference towards either of the firms buy a long-term contract of the preferred firm. Customers with moderate preference towards either of the firm buy a short-term contract of the preferred firm. The main point of the paper is to determine how the presence of customer lock-in affects profits and consumer welfare. The basis of comparison is the case where long-term contracts are not allowed. With only short-term contracts, there is a unique, symmetric solution, in which the market is split equally in both periods. One could argue that when long-term contracts are also allowed, price discrimination becomes more effective, because it breaks the market into four

\(^2\)According to Fudenberg and Tirole, this kind of price discrimination does not fit into the textbook classification of first-, second- or third-degree (originating from Pigou). In our opinion, on the contrary, this is a sort of third-degree price discrimination, because one could view the first-period choice as an observable characteristic of the customer, much like being a student or a senior citizen.
segments instead of just two. Thus, one would be tempted to say that more effective price discrimination benefits firms and hurts consumers. However, this is not always the case here. Long-term prices have to be set lower than short-term prices, otherwise nobody would buy long-term contracts. When all prices are determined competitively, firms have to compensate consumers for the fact that they are locked-in.

As far as how industry structure is concerned, the two closest papers to our work are Caminal and Matutes (1990) and Fudenberg and Tirole (2000). Both works feature that in equilibrium some customers switching firms from one period to the other. However, the driving force that induces customers to switch are quite different in the two models. In Fudenberg and Tirole (2000), it is due to poaching, whereas in Caminal and Matutes (1990) it happens because consumers’ tastes change over time.\(^3\) In this respect, we follow the assumption of Caminal and Matutes (1990). However, our way of modeling customer preferences is substantially more general though, because Caminal and Matutes (1990) consider only the case where types are independent across periods. We allow for correlation between types in the two periods (i.e. intertemporal correlation). In fact, our major interest is to examine how this correlation coefficient affects the equilibrium outcome.

Unlike Fudenberg and Tirole (2000), we do not allow for behavioral-based price discrimination. Our model focuses solely on second-degree price discrimination. This is quite a recent line of research. At its most general level, it was studied by Stole (1995) and Rochet and Stole (1999). They show how non-linear price schedules are determined in duopoly. A special case of this problem is the one, in which the offered menu consists of only two elements: a long-term contract and a short-term contract. In this setting, consumers are screened into four different sets, corresponding to each of the offered contracts. Incentive compatibility typically implies that these sets will be consecutive intervals. Fudenberg and Tirole (2000) are who first study this problem.\(^4\)

The reason why we have chosen not to allow for behavioral-based price discrimination, was more for the

\(^3\)Hence, at Caminal and Matutes switching is efficient, because customers always choose their most preferred firm. In fact, Fudenberg and Tirole (2000) examine changing preferences in their last section too, but it is not the main focus of their paper.

\(^4\)A drawback of their model is that there remains an indeterminacy between long-term and short-term contracts of a given firm. In other words, in their model all customers have the same preference ordering over the two types of contracts of a given firm. Therefore, either all customers strictly prefer one of the contracts, or they are all indifferent. In equilibrium, this latter occurs. Therefore the segmentation of customers into four intervals is supported by an ad-hoc assumption. They assume that among customers who choose a particular firm, those who are located closer to the firm will choose long-term contracts, whereas the rest will choose short-term contracts, even though all of them are indifferent. The need for this “tie-breaking” assumption is due to the deterministic structure of the model, i.e. that customers’ preferences are fixed. We overcome this problem by imposing a stochastic structure: we allow customers’ preferences to change from one period to another. As a consequence, we get perfect separation based on the type of the consumers: almost all of them will have a strictly preferred choice except, of course, the marginal customer. Indeed, Fudenberg and Tirole (2000) mention in footnote 20, that the indeterminacy could be solved this way, but they do not pursue this line.
sake of simplicity than for any conceptual reason.\textsuperscript{5} Our analysis would carry through, although everything would be algebraically more complicated. As a matter of fact, in Fudenberg and Tirole (2000) firms employ a mix of two price discriminatory practices, which have opposing effects. As we have mentioned it earlier, offering long-term contracts makes switching less likely, whereas poaching makes it more likely. We focus only on the effects of the first (second-degree price discrimination) and do not want our results to be garbled by the presence of poaching.

For the most part, we pose the same questions as Fudenberg and Tirole (2000). Hence, some of our results are comparable with some of theirs. First, they derive that offering a long-term contract is a credible commitment to more aggressive pricing in the second period. Since we make a more general distributional assumption, we can make this statement more precise. The validity of this result depends on the intertemporal correlation of types. For highly correlated types\textsuperscript{6}, firms indeed charge lower than static prices in the second period. Inversely, if correlation between types is low, we get the opposite result. Second, they show that the presence of long-term contracts increases efficiency, because there will be less switching. This result is solely due to the possibility of poaching. This practice makes switching inefficient. We do not allow for poaching, therefore if customers switch, they do it because (were prices equal) they like the other firm more. Thus, the possibility of switching increases efficiency. Hence, we get the opposite result, namely, nothing can be more efficient than banning long-term contracts. Third, they show that the ability to use long-term contracts creates a prisoners’ dilemma-like situation. If firms are allowed to offer both types of contracts, it is a dominating strategy to use both of them. However, prices and profits are lower relative to the case where firms use only short-term contracts. Again, we confirm this result, but only conditionally. The statement is true if intertemporal correlation of types is high, otherwise we get the opposite result. Fourth, Caminal and Matutes (1990) investigate the other extreme, where types are independent. They show that using only long-term contracts is worse than using only short-term contracts.\textsuperscript{7} For small intertemporal correlation of types, we confirm this finding.

\section{The model}

We set up a symmetric model of two firms. For simplicity, we assume that firms have zero cost. There are two time-periods. We assume that neither consumers, nor firms discount the future. We consider the circular city model, in which consumers are distributed along a circle. The two firms, $A$ and $B$ are\textsuperscript{5} Although we do think that poaching is more typical in industries that deliver services directly to the consumers’ residence. Cellular phone service does not fit this description, of course.\textsuperscript{6} Note that Fudenberg and Tirole (2000) consider fixed types, so there the correlation coefficient is 1.\textsuperscript{7} Strictly speaking, there are no long-term contracts in Caminal and Matutes (1990). However, it can be shown that precommitment to a second-period price is equivalent to a long-term contract.
positioned at the endpoints of a diameter. Consumers bear transportation costs that are proportional to their distance from the firms (linear). The cost of traveling a unit distance is \( t \). In the second period, every consumer gets an independent, and identically distributed taste shock. This taste shock reflects the fact that consumers’ relative preferences towards A and B could change over time. The i.i.d. assumption guarantees that the distribution is invariant, that is, the (unconditional) second-period distribution is the same as the first-period distribution. To keep matters simple, we make the following additional distributional assumptions.

**Assumption 1.** Consumers are uniformly distributed along the circle in both periods.

**Assumption 2.** The taste shock is a zero mean, symmetrically distributed random variable, with pdf \( \varphi(\cdot) \), cdf \( \Phi \).

These two assumptions guarantee that the problem is symmetric, that is, nothing would change if the two firms switched positions. It will be convenient to introduce the conditional second-period distribution, \( f(x_2 | x_1) \). Let the circumference of the circle be normalized to 2, and let A and B’s position be 0 and 1, respectively. Without loss of generality, we can assume that the support of \( \varphi(\cdot) \) is \([-1, 1]\). The reason is that since the circumference is 2, the largest distance between a customer’s old and new position is 1.

Given the above assumptions, it is easy to derive the following consequences:

**Consequence 1.** Symmetry

\[
 f(x_2 | x_1) = \varphi(x_2 - x_1) = \varphi(x_1 - x_2) = f(1 - x_2 | 1 - x_1)
\]

**Consequence 2.** Conditional mean

\[
 E[x_2 | x_1] = x_1
\]

In addition, assume that the value of the service (reservation value), \( v \) is identical across consumers, and it is high enough that all consumers find it worthwhile to buy from one of the firms in both periods. As a consequence, consumers only care about price and transportation costs.

Firms can offer short-term and long-term contracts. As a shorthand, we will use the notations: \( LT_A \), \( ST_A \), \( ST_B \), and \( LT_B \). If a consumer chooses a long-term contract then he has to stay with this firm in

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8 The simplest way to model product differentiation is with customers distributed along the unit segment, à la Hotelling. However, in this case, the type-distribution would not be invariant. In fact, the second-period distribution would be heavier on the tails than the first-period distribution. This fact would greatly influence the results. In particular, if we repeated a one-shot game twice, we would get higher prices in the second period than in the first, only because the distribution has changed in this particular way. Since we do not want our results be driven by changes in the distribution of customers, we consider the circular model. We are grateful to Guido Menzio for suggesting this.

9 This assumption simplifies the analysis to a great deal. As usual in Hotelling-models, the main results would still hold with non-trivial reservation values, although the algebra would be a lot more complicated.
both periods. On the other hand, if a consumer chooses a short-term contract, then he can choose again in the second period whichever firm he likes most. Let the price of the long-term contract be \( l_i \). Let the prices of the short-term contracts be \( f_i \) (first) and \( s_i \) (second). In principle, the firms could offer different long-term prices in the two periods, but since all that the consumers care about is the sum of the two, we make the simplifying assumption that they are the same across periods. In both periods, first firms set prices simultaneously, then customers make their choices, also simultaneously.

We will characterize the equilibrium of this game. The equilibrium concept that we use is sequential equilibrium (or rational expectations equilibrium). We require subgame perfectness from the solution. In the present context, it means that the prices \( s_A \) and \( s_B \) must be equilibrium outcomes of the subgame that firms play in the second period. In other words, firms cannot commit to their second-period prices in the first stage. Also, in equilibrium, both firms and customers correctly anticipate what prices will be set in the second period. Below we formalize this concept.

**Definition 1** The sequential (or rational expectations) equilibrium of the game is a set of prices \( \{l_i, f_i, s_i\}_{i=A,B} \) and choice functions \( \sigma_1(x_1), \sigma_2(x_2) \in \{LT_A, ST_A, ST_B, LT_B\} \) such that the following conditions are met:

**Second period:**

i) Consider a customer of type \( x_2 \). If he chose either \( LT_A \) or \( LT_B \) in the first period then he must choose the same contract. Otherwise, given \( s_A \) and \( s_B \), his best choice is \( \sigma_2(x_2) \).

ii) Given the first-period prices \( \{l_i, f_i\}_{i=A,B} \), customers’ observed first-period choices, and customers’ (correctly) anticipated second-period behavior, \( \sigma_2(x_2) \), \( s_A \) and \( s_B \) are the equilibrium outcome of the subgame, in which both firms try to maximize their respective second-period profits.

**First period:**

iii) Consider a customer of type \( x_1 \). Given prices \( \{l_i, f_i\}_{i=A,B} \), other customers’ (correctly) anticipated second-period behavior, \( \sigma_2(x_2) \), the (correctly) anticipated second-period prices \( s_A \) and \( s_B \), and one’s own conditional type distribution \( F(x_2|x_1) \), the best choice of this customer is \( \sigma_1(x_1) \).

iv) Given customers’ (correctly) anticipated behavior \( \sigma_1(x_1) \) and \( \sigma_2(x_2) \), and the fact that \( s_A \) and \( s_B \) are going to be subgame-perfect prices, \( \{l_i, f_i\}_{i=A,B} \) are the equilibrium outcome of the game, in which both firms try to maximize their respective total profits.

We focus attention on symmetric equilibria where firms’ strategies are the same. Even before going into the algebraic analysis, we briefly characterize the three possible types of symmetric equilibria:

1. The “short-term contracts only” equilibrium. Firms may or may not offer long-term contracts. In either case, customers buy only short-term contracts. Customers’ choices are the following:

\[
\sigma_t(x_t) = \begin{cases} 
ST_A & \text{if } 0 \leq x_t \leq \frac{1}{2} \\
ST_B & \text{if } \frac{1}{2} \leq x_t \leq 1 
\end{cases}
\]
Prices are equal: \( f_A = f_B \) and \( s_A = s_B \).

This equilibrium is an efficient outcome, because customers always buy from the “closest” firm, so transportation costs are minimized.

2. The “both types of contracts” equilibrium. Firms offer both types of contracts, and sell positive amounts of both of them. Customers choices are the following:

\[
\sigma_1(x_1) = \begin{cases} 
LT_A & \text{if } 0 \leq x_1 \leq A \\
ST_A & \text{if } A \leq x_1 \leq \frac{1}{2} \\
ST_B & \text{if } \frac{1}{2} \leq x_1 \leq B \\
LT_B & \text{if } B \leq x_1 \leq 1 
\end{cases} \\
\sigma_2(x_2, \sigma_1(x_1)) = \begin{cases} 
LT_A & \text{if } \sigma_1(x_1) = LT_A \\
ST_A & \text{if } \sigma_1(x_1) \in \{ST_A, ST_B\}, \text{ and } 0 \leq x_2 \leq \frac{1}{2} \\
ST_B & \text{if } \sigma_1(x_1) \in \{ST_A, ST_B\}, \text{ and } \frac{1}{2} \leq x_2 \leq 1 \\
LT_B & \text{if } \sigma_1(x_1) = LT_B
\end{cases}
\]

Prices are equal: \( l_A = l_B, f_A = f_B \) and \( s_A = s_B \). The cutoffs are symmetric, too: \( B = 1 - A \).

Figure 1: Customers’ first-period choice in the “both types of contracts” equilibrium

This equilibrium produces an ex-post inefficient outcome, because long-term contractors do not always
buy from the “closest” firm. Those, whose taste shock turns out to be sufficiently unfavorable, should
switch firms, but they are not allowed to do that if they have bought long-term contract in the first
period. Therefore, transportation costs are not minimized.
In this type of equilibrium it also must be true that:

\[ 2l_i \leq f_i + s_i \]

If this condition does not hold, then no customer would choose a long-term contract. In other words,
firms have to offer long-term contracts at a discount, in order that any customer accepts to be locked-in.

3. The “long-term contracts only” equilibrium. Firms may or may not offer short-term contracts. In
either case, customers buy only long-term contracts. Customers’ choices are the following:

\[
\sigma_1 (x_1) = \sigma_2 (x_2, \sigma_1 (x_1)) = \begin{cases} 
LT_A & \text{if } 0 \leq x_1 \leq \frac{1}{2} \\
LT_B & \text{if } \frac{1}{2} \leq x_1 \leq 1 
\end{cases}
\]

Prices are equal: \( l_A = l_B \). This equilibrium also produces an ex-post inefficient outcome, for the
same reason than in the previous case.

In general, it depends on the shock-distribution, which of the above three possible symmetric solutions
will occur. Unfortunately, we cannot provide general conditions that guarantee the existence of each of
these equilibria. In Section 3, we assume that the shock-distribution is uniform. Then we will be able to
state sufficient conditions for the existence. As we will see in that example, for all parameter values there
will be at least one symmetric equilibrium. In addition, for certain parameter values there will be two.

In the subsections that follow, we present an approach to solve each of these cases at the general level.

2.1 “Short-term contracts only”

If no customer buys long-term contracts then there is no intertemporal linkage between the two periods.
Therefore, customers buy in each period the contract which comes at the lowest cost for them. In addition,
our assumptions about the shock distribution guarantee that customers are uniformly distributed along
the circle in both periods. It is straightforward to show that prices will be equal to the transportation cost,
as in the standard Hotelling-model.

\[ f_A = f_{B} = s_A = s_{B} = t \]
2.2 “Both types of contracts”

2.2.1 Consumers’ second-period choice

In the second period, only those consumers have a choice who did not commit themselves (to long-term contracts) in the first period. As in the textbook model, there will be a cutoff value, $S$ such that the customer at $S$ is indifferent between the two firms, while everyone to the left of $S$ strictly prefer $A$, and vice versa. It is well-known that $S$ will be given as:

$$S = \frac{1}{2} + \frac{s_A + s_B}{2t}$$

2.2.2 Consumers’ first-period choice

To cut down the number of cases, we will impose an additional restriction on the shock. In particular, we assume that the support of its distribution is $[-k/2, k/2]$, instead of $[-1, 1]$. $k$ is assumed to be on one hand “large enough,” so that $A + \frac{k}{2} \geq \frac{1}{2}$, but “small enough,” so that $A + \frac{k}{2} \leq 1$ and $\frac{k}{2} \leq A + \frac{1}{2}$. The first assumption is necessary, in order that both short-term and long-term contracts be sold in equilibrium. The last two conditions are not crucial, the analysis would go through even if they did not hold, but the algebra would be more complicated. Since we only want to present the approach that leads to the solution, we have chosen to restrict attention to this simplest case.

Consider a customer on the upper half of the circle, that is an $x_1 \in [0, 1]$. The analysis for the customers at the opposite half ($x_1 \in [1, 2]$) goes along the same lines. Since the shock is in the interval $[-k/2, k/2]$, the second-period type, $x_2$ is in the set $[x_1 - k/2, x_1 + k/2]$.

The expected cost of contract $LT_A$ for a customer located at $x_1$ is:

$$P_{LT_A}(x_1) = 2l_A + t \left( x_1 + \int_{x_1-k/2}^{0} -x_2 f(x_2 \mid x_1) \, dx_2 + \int_{0}^{x_1+k/2} x_2 f(x_2 \mid x_1) \, dx_2 \right) \quad (1)$$

The expected cost of contract $ST_A$ for a customer located at $x_1$ is:

$$P_{ST_A}(x_1) = f_A + tx_1 + s_A F(S \mid x_1) + s_B \left( 1 - F(S \mid x_1) \right) + t \left( \int_{x_1-k/2}^{0} -x_2 f(x_2 \mid x_1) \, dx_2 + \int_{0}^{S} x_2 f(x_2 \mid x_1) \, dx_2 + \int_{S}^{x_1+k/2} (1 - x_2) f(x_2 \mid x_1) \, dx_2 \right) \quad (2)$$

In period 1, a customer would prefer $LT_A$ over $ST_A$ if and only if

$$P_{LT_A}(x_1) - P_{ST_A}(x_1) \leq 0$$

Substituting (1) and (2) yields
After integration by parts, this formula simplifies:

\[ 2l_A - f_A - s_A F(S|x_1) - s_B (1 - F(S|x_1)) + t \int_{S}^{x_1+k/2} (2x_2 - 1) f(x_2|x_1) \, dx_2 \leq 0 \]

This expression is a strictly increasing function of \( x_1 \). To show this, notice that \( \frac{\partial}{\partial x_1} F(x_2|x_1) = -\varphi(x_2 - x_1) = -f(x_2|x_1) \). Also, \( F(x_1 + k/2|x_1) = 1 \). Using these facts, the derivative of (3) simplifies to:

\[ 2t (1 - F(S|x_1)) \]

This term is always positive, so (3) is indeed increasing in \( x_1 \). The threshold level \( A \) will be such that \( P_{LT_A}(A) - P_{ST_A}(A) = 0 \). Formally,

\[ g^A = 2l_A - f_A - s_A + 2t \int_{S}^{A+k/2} (1 - F(x_2|A)) \, dx_2 = 0 \]

(4)

\( A \) is the location of the marginal customer, who is indifferent between the contracts \( LT_A \) and \( ST_A \). Customers to the left of \( A \) prefer \( LT_A \), customers to the right prefer \( ST_A \) (see Figure 1).

The derivation of cutoff \( B \) involves essentially the same steps. Hence we only present the result.

\[ g^B = 2l_B - f_B - s_B + 2t \int_{S}^{B-k/2} F(x_2|B) \, dx_2 = 0 \]

(5)

This expression is a decreasing function of \( B \), meaning that customers above this threshold will choose the contract \( LT_B \), whereas customers below choose \( ST_B \) (see Figure 1).

Finally, we determine cutoff \( C \). In period 1, a customer would prefer \( ST_A \) over \( ST_B \) if and only if

\[ P_{ST_A}(x_1) \leq P_{ST_B}(x_1) \]

This reduces to:

\[ f_A + tx_1 \leq f_B + t (1 - x_1) \]

Rearranging:

\[ x_1 \leq \frac{1}{2} + \frac{-f_A + f_B}{2t} \equiv C \]

(6)

In equilibrium, if all four contracts are sold, it must be true that
Thus, consumers segment themselves into four categories in the first period. Those, who strongly prefer one of the firms opt for a long-term contract of the preferred firm. These customers are quite certain that they will not want to switch in the second period, therefore they can take the risk of locking themselves into a long-term contract. Those, who only moderately prefer one of the firms choose a short-term contract. They are more likely to switch, therefore they want to avoid to get locked-in.

2.2.3 Firms’ choice

We look for symmetric equilibrium, therefore we only display firm A’s conditions. Since we require subgame-perfection, we first solve for the firms’ second-period spot prices. In period 2, firms compete in second-period prices, taking the first-period prices and consumer choices as given. Hence, at the time of choosing the second-period prices, the first-period cutoff values (A, B and C) are already determined. Therefore, firms are not taking into account the effect that these prices have on the cutoffs.\(^{10}\)

The second-period profits are:

\[
\Pi_A^2 = l_A A + s_A \int_A^B F(S|x_1) \, dx_1 \\
\Pi_B^2 = l_B (1 - B) + s_B \int_A^B (1 - F(S|x_1)) \, dx_1
\]

The first-order conditions with respect to second-period prices are the following:

\[
\frac{\partial \Pi_A^2}{\partial s_A} = \int_A^B F(S|x_1) \, dx_1 - \frac{s_A}{2} \int_A^B f(S|x_1) \, dx_1 \tag{7}
\]

\[
\frac{\partial \Pi_B^2}{\partial s_B} = \int_A^B (1 - F(S|x_1)) \, dx_1 - \frac{s_B}{2} \int_A^B f(S|x_1) \, dx_1 \tag{8}
\]

The first-period profit:

\[
\Pi_A^1 = l_A A + f_A (C - A)
\]

Since we assume no discounting, firm A just maximizes the sum of its profits:

\[
\Pi_A = \Pi_A^1 + \Pi_A^2
\]

\(^{10}\)Things would be different if firms could credibly commit to \(s_A\) and \(s_B\) in the first stage. Then we would need to take into account the effect that these prices have on \(A, B\) and \(C\).
Let us now derive the conditions that determine first-period prices. Other than having a direct effect, first-period prices have several indirect effects. On one hand, they influence the first-period cutoff values \((A \text{ and } B)\).\(^{11}\) On the other hand, they also influence the equilibrium outcome of the second period \((s_A, \text{ and } s_B)\). Formally, the two first-order conditions expressed in matrix form are the following:

\[
\begin{bmatrix}
\frac{dl_A}{dl_A} & \frac{dl_A}{df_A} \\
\frac{dl_A}{df_A} & \frac{dl_A}{df_A}
\end{bmatrix}
+ \begin{bmatrix}
\frac{dl_A}{dl_A} & \frac{dl_A}{df_B} & \frac{dl_A}{ds_A} & \frac{dl_A}{ds_B} \\
\frac{dl_A}{df_B} & \frac{dl_A}{df_B} & \frac{dl_A}{ds_A} & \frac{dl_A}{ds_B} \\
\frac{dl_A}{ds_A} & \frac{dl_A}{ds_A} & \frac{dl_A}{ds_A} & \frac{dl_A}{ds_B} \\
\frac{dl_A}{ds_B} & \frac{dl_A}{ds_B} & \frac{dl_A}{ds_B} & \frac{dl_A}{ds_B}
\end{bmatrix}
= \begin{bmatrix}
\frac{dA}{dl_A} & \frac{dA}{df_A} \\
\frac{dB}{dl_A} & \frac{dB}{df_A} \\
\frac{ds_A}{dl_A} & \frac{ds_A}{df_A} \\
\frac{ds_B}{dl_A} & \frac{ds_B}{df_A}
\end{bmatrix}
\]

(9)

It is straightforward to express the partial derivatives of \(\Pi_A\):

\[
\frac{\partial \Pi_A}{\partial l_A} = 2A
\]

\[
\frac{\partial \Pi_A}{\partial f_A} = (C - A) - \frac{f_A}{2t}
\]

\[
\frac{\partial \Pi_A}{\partial A} = 2l_A - f_A - s_A f(S|A)
\]

\[
\frac{\partial \Pi_A}{\partial B} = s_A f(S|B)
\]

\[
\frac{\partial \Pi_A}{\partial s_A} = 0
\]

\[
\frac{\partial \Pi_A}{\partial s_B} = \frac{s_A}{2t} \int_A^B f(S|x_1) dx_1
\]

(10)

To find out the indirect terms (the elements of the last matrix in (9)), notice that \(A, B, s_A\) and \(s_B\) are jointly determined by the equations (4), (5), (7) and (8). So differentiate totally the system of these four equations:

\[
\begin{bmatrix}
\frac{\partial g_A}{\partial l_A} & \frac{\partial g_A}{\partial f_A} \\
\frac{\partial g_A}{\partial f_A} & \frac{\partial g_A}{\partial f_A}
\end{bmatrix}
+ \begin{bmatrix}
\frac{\partial g_A}{\partial l_A} & \frac{\partial g_A}{\partial B} & \frac{\partial g_A}{\partial s_A} & \frac{\partial g_A}{\partial s_B} \\
\frac{\partial g_A}{\partial B} & \frac{\partial g_A}{\partial B} & \frac{\partial g_A}{\partial s_A} & \frac{\partial g_A}{\partial s_B} \\
\frac{\partial g_A}{\partial s_A} & \frac{\partial g_A}{\partial s_A} & \frac{\partial g_A}{\partial s_A} & \frac{\partial g_A}{\partial s_B} \\
\frac{\partial g_A}{\partial s_B} & \frac{\partial g_A}{\partial s_B} & \frac{\partial g_A}{\partial s_B} & \frac{\partial g_A}{\partial s_B}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

Rearranging this matrix equation yields the aforementioned indirect terms:

\(^{11}\text{We did not mention the indirect effect of the prices through cutoff } C. \text{ In fact, } C \text{ is given by a fairly simple expression, as seen in (6). It is a linear function of } f_A \text{ and it does not depend on } l_A. \text{ Therefore, we incorporate this indirect effect of } f_A \text{ into the direct effect of } f_A. \text{ This will be clear by looking at equation (10) later.}\)
Proposition 1  The system (9) has a symmetric interior solution, provided that the following conditions are met.
i) 
\[ 1 + 4 \int_0^1 (1 - \Phi(\varepsilon)) d\varepsilon > \frac{1}{\varphi(0)} \]

ii) Either
\[ h > 2 \frac{-8H^3 + 16H^2 - 4H - 1}{7 - 2H} \]
or
\[ h < H (-4H^2 + 8H - 3) \]

where \( h = \varphi(1/2), \ H = \Phi(1/2) \)
Proof. Appendix. ■

These conditions are neither simple algebraically, nor very intuitive. Nevertheless, it is relatively simple to check for any parametric distribution whether they hold or not.

Unfortunately, at this level of generality there is no guarantee for the unicity of the solution. There is no guarantee, in general, that the problem is globally concave. More importantly, even if the problem was globally concave, and thus there was a unique global maximizer, we should still check for corner-type deviations. It turns out that firms could deviate, by offering only one type of contracts (short-term or long-term). These are what we will call “corner-type” deviation strategies later. We must check whether or not these deviation strategies are more profitable than the proposed equilibrium strategy.

We are only able to express one of the terms, provided that there exists an interior solution, i.e. a root to the first-order conditions. We do this in the following Proposition.

**Proposition 2** Suppose that there exists a symmetric (interior) solution to the “both types of contracts” problem, that is the system (9) has positive roots. Then $f_A = t$.

**Proof.** The proof is straightforward. Substitute the direct terms $\left( \frac{\partial \Pi}{\partial A} \right.$ and $\frac{\partial \Pi}{\partial f_A}$) into the first-order conditions.

\[
\frac{d\Pi_A}{dA} = 2A - \left[ \frac{\partial \Pi}{\partial A} \frac{\partial \Pi}{\partial A} \frac{\partial \Pi}{\partial A} \frac{\partial \Pi}{\partial A} \right] \Omega^{-1} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0
\]

\[
\frac{d\Pi_A}{df_A} = (C - A) - \frac{f_A}{2} - \left[ \frac{\partial \Pi}{\partial A} \frac{\partial \Pi}{\partial A} \frac{\partial \Pi}{\partial A} \frac{\partial \Pi}{\partial A} \right] \Omega^{-1} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0
\]

Express the second term of the first equation and substitute into the second equation:

\[
\frac{d\Pi_A}{df_A} = (C - A) - \frac{f_A}{2} + A = 0
\]

Since in a symmetric equilibrium, $C = \frac{1}{2}$, we get that $f_A = t$.

The result that we have obtained here is that the first-period price coincides with the equilibrium price of the static Hotelling model.

15
2.3 “Long-term contracts only”

Since customers have only two options, there will be only one cutoff, $C$. No customer is allowed to switch in the 2nd period. Customers choose $LT_A$ if $0 \leq x_1 \leq C$, otherwise they choose $LT_B$. $C$ is the location of the customer who is indifferent between the two contracts. The expected costs of these contracts for the marginal customer are the following:

$$
P_{LT_A} = 2l_A + tC + t \left( \int_{C-1}^{0} -x_2 f' (x_2 | C) dx_2 + \int_{1}^{1} x_2 f (x_2 | C) dx_2 + \int_{1}^{C+1} (2 - x_2) f (x_2 | C) dx_2 \right)$$

$$
P_{LT_B} = 2l_B + t (1 - C) + t \left( \int_{C-1}^{0} (x_2 + 1) f (x_2 | C) dx_2 + \int_{1}^{1} (1 - x_2) f (x_2 | C) dx_2 + \int_{1}^{C+1} (x_2 - 1) f (x_2 | C) dx_2 \right)
$$

Subtracting $P_{LT_B}$ from $P_{LT_A}$, after integration by parts yields the following:

$$
g^C = P_{LT_A} - P_{LT_B} = 2 (l_A - l_B) + 2t \left( 1 - 2 \int_{0}^{1} F (x_2 | C) dx_2 \right) = 0 \quad (12)
$$

The profit of firm $A$ is $\Pi_A = 2l_A C$, hence the first-order condition:

$$
\frac{\partial \Pi_A}{\partial l_A} = 2C + 2l_A \frac{\partial C}{\partial l_A} = 0
$$

Total differentiation of (12) yields the indirect term:

$$
\frac{\partial C}{\partial l_A} = \frac{2}{4t \int_{0}^{1} \frac{\partial}{\partial x_1} F (x_2 | C) dx_2} = -\frac{1}{2t} \frac{1}{F (1 | C) - F (0 | C)}
$$

In the second equality we used the fact that $\frac{\partial}{\partial x_1} F (x_2 | x_1) = -\varphi (x_2 - x_1) = -f (x_2 | x_1)$. Also, in a symmetric equilibrium, $C = \frac{1}{2}$. Thus, substituting $\frac{\partial C}{\partial l_A}$ into the first-order condition, and solving it yields:

$$
l_A = t \left( F (1 | 1/2) - F (0 | 1/2) \right) = t \left( 2 \Phi (1/2) - 1 \right)
$$

3 Example: Uniformly distributed shock

In the previous section we have provided an approach to solving for the equilibrium of the game. The two simple cases (in which firms offer only one kind of contract) are globally concave maximization problems. Therefore, they yield a unique solution, and we were also able to establish them explicitly. The most complex case (“both types of contracts”) is not globally concave, in general. We could state some conditions that guarantee existence, if the solution is interior. However, there may be multiple interior solutions. Moreover, in order to establish an equilibrium, we must check whether the proposed interior equilibrium holds against corner-type deviation strategies. This must be done not only in the “both types of contracts”
regime, but in all three regimes. For instance, in the first regime (“short-term only”) we should check whether a firm has a profitable deviation by offering both types of contracts, or offering only long-term contracts. In full generality, it is extremely hard even to establish formally these conditions. Moreover, what we would really like to find is not these conditions per se, but the restrictions that are needed to be imposed on the shock-distribution so that we get these conditions to hold. This seems an impossible task for us.

One could, of course, take a numerical approach, by assuming particular parametric distributions in place of \( f (\cdot | \cdot) \), compute the solutions, and check them against the “corner-type” deviation strategies.

In this section, we consider the simplest parametric specification of the shock-distribution. Interestingly, we get closed form solutions for all three regimes. Moreover, we are able to state sufficient (though not always sharp) conditions for each types of symmetric equilibria.

In particular, assume a uniformly distributed taste shock over the interval \([-k/2, k/2]\). Hence, the conditional distribution of second-period types takes a particularly simple functional form:

\[
\begin{align*}
    f (x_2 | x_1) &= \begin{cases} 
    0 & \text{if } x_2 < x_1 - \frac{k}{2} \\
    \frac{1}{2} & \text{if } x_1 - \frac{k}{2} \leq x_2 \leq x_1 + \frac{k}{2} \\
    0 & \text{if } x_1 + \frac{k}{2} < x_2 
    \end{cases} \\
    F(x_2 | x_1) &= \begin{cases} 
    0 & \text{if } x_2 < x_1 - \frac{k}{2} \\
    \frac{x_2 - x_1}{k} + \frac{1}{2} & \text{if } x_1 - \frac{k}{2} \leq x_2 \leq x_1 + \frac{k}{2} \\
    1 & \text{if } x_1 + \frac{k}{2} < x_2 
    \end{cases}
\end{align*}
\]

In words, if a customer’s original position was \( x_1 \) then in the second period he will be located in the interval \([x_1 - k/2, x_1 + k/2]\). Over this interval, all locations are equally likely. The unique parameter, \( k \), represents the spread (or indirectly the variance) of the distribution. We restrict attention to values of \( k \) within the interval \([0, 2]\). The two extremes represent special cases. When \( k = 0 \), then the shock is zero, so we have perfect correlation between first-period type and second-period type (in fact, they are the same). When \( k = 2 \), then first-period type and second-period type are independent, so there is zero correlation between them.

We have the following three results.

**Proposition 3** If \( 0 \leq k \leq 1 \) then there exists a symmetric equilibrium, in which firms offer only short-term contracts. The equilibrium prices are the following:

\[
\begin{align*}
    f_A &= f_B = t \\
    s_A &= s_B = t
\end{align*}
\]
In addition, these bounds are sharp, i.e. when $1 < k \leq 2$, then the above strategies do not constitute an equilibrium.

**Proof.** Appendix. ■

In this regime, $k$ does not affect equilibrium prices, so it does not affect profits either. The reason is, of course, that there is no intertemporal linkage between the two periods. Customers choose in each period their least costly option. In making their choice in the first period, they do not care about their second-period types, and vice versa.

**Proposition 4** If $0.81 \leq k \leq 1.28$ then there exists a symmetric equilibrium, in which firms offer both short-term and long-term contracts. The equilibrium prices and cutoffs are the following:

\[
\begin{align*}
l_A &= l_B = \frac{79k^2 + 164k - 4 + (2 - k)d}{256k} \\
f_A &= f_B = t \\
s_A &= s_B = kt \\
A &= \frac{6 - 7k + d}{16} \\
B &= \frac{10 + 7k - d}{16}
\end{align*}
\]

where $d = \sqrt{(97k^2 - 68k + 4)}$

**Proof.** Appendix. ■

In this regime, long-term prices and second-period spot prices increase in $k$ (see Figure 2), and consequently, so do profits. Also, the cutoff $A$ is increasing in $k$ (see Figure 3), meaning that as the variance of the shock gets larger, more customers would buy long-term contracts instead of short-term contracts.

For an intuitive explanation of the result $s_A = s_B = kt$, consider customers with $A \leq x_1 \leq B$. These are the only customers whom firms target with prices $s_A$ and $s_B$ in the second period. Although the second-period distribution of all customers is uniform, the distribution of this select group of customers is not. This distribution has, in fact, a trapezoid-shaped pdf. As $k$ increases, more and more weight will be put on the tails of this distribution. With more uncertainty, the taste shock relocates middle customers more towards the extremes, on average. It is more favorable for firms if the distribution is “heavier” on

---

ootnote{These bounds are only approximate. The exact lower bound is the root of the following 8th order polynomial: $38046501Z^8 - 80585340Z^7 - 9011246Z^6 + 63690268Z^5 - 1313407Z^4 - 11634304Z^3 - 2170424Z^2 - 127232Z - 2352 = 0$. The exact upper bound is $\frac{1}{4} + \frac{\sqrt{15}}{4}$.}
the tails, because it means more differentiation. As a result, equilibrium prices will be higher. It turns out, that with uniformly distributed shocks, the second-period prices are exactly proportional to $k$.

With $f_A$ and $f_B$ constants and $s_A$ and $s_B$ increasing in $k$, it must be that $l_A$ and $l_B$ are also increasing in $k$. The reason is that a long-term contract is a closer substitute with the short-term contracts than with the long-term contract of the other firm. Therefore, if the total price (first-period price plus some weighted average of the two second-period prices) of the short-term contracts increases, then firms can increase the prices of the long-term contracts, too.

Our third result, the fact that the cutoff $A$ is increasing in $k$, is probably the most difficult to anticipate. In fact, a priori one could even reach the opposite conclusion, by arguing that more uncertainty would induce customers to get the safer option, that is, the short-term contract. Probably, this argument would hold true in a monopoly context, though it does not in our duopoly setting. It is certainly true that, ceteris paribus (holding all prices fixed), an increase in uncertainty would make long-term contracts less attractive for all customers. However, when setting prices, firms adjust for the increased uncertainty by changing prices. In fact, they increase $s_A$ and $s_B$ by more than $l_A$ and $l_B$, as one can see on Figure 2. This change in relative prices more than offsets the effect of increased uncertainty.

**Proposition 5** If $1.25 \leq k \leq 2$, then there exists a symmetric equilibrium, in which firms offer only long-
equilibrium as a function of $k$ in the “both types of contracts” regime ($t$ is normalized to 1)

**term contracts.** The equilibrium prices are the following:

$$l_A = l_B = \frac{t}{k}$$

**Proof.** Appendix.

In this regime, prices decrease in $k$, and so do profits. An intuitive explanation is the following. In this regime, there is no switching. Every customer that a firm can attract stays with the firm for both periods. This fact induces a more fierce price competition than the one that occurs in a one-shot game. In other words, firms are more eager to lower prices, since the gain from lowering prices is doubled. Now, intertemporal correlation between types mitigates the effect of a price cut. To see this, consider a customer on the turf of $A$ (that is, $x_1 < 1/2$). Since we assume a zero-mean shock, this customer will be always more likely to be closer to $A$ in the second-period too, regardless of the variance of the shock. However, the higher the intertemporal correlation between types, the greater the probability that this customer will be closer to $A$ than to $B$. Therefore, this customer would be less and less tempted to switch to firm $B$, had $B$ cut its price. Consequently, higher intertemporal correlation (smaller $k$) results in less fierce price competition. Inversely, as $k$ increases, there is less and less intertemporal dependence between types, and price competition intensifies, leading to lower prices.
As one can see, we have identified existence regimes for each type of equilibria. Only two of these bounds are not sharp. On one hand, we were not able to prove that for $1.28 < k \leq 2$, there is no “both types of contracts” equilibrium. On the other hand, there might be “long-term only” equilibrium for $k < 1.25$. In both cases, we did numerical calculations, that have shown us that the sharp bounds are very close to the ones we stated above, but it is cumbersome to express them analytically.

We can learn the following conclusions from this example. First, the variance of the shock (the correlation between first-period and second-period types) influences, which type of equilibrium will occur. For small shocks (high correlation between types), both firms offer only short-term contracts. For medium shocks (medium correlation between types), firms offer both types of contracts. For large shocks (low correlation between types), both firms offer only long-term contracts. To be precise, in this example, there is some overlap between these regimes. In particular, for $0.81 \leq k \leq 1$, both “short-term only” and “both types” equilibrium exists. Also, for $1.25 \leq k \leq 1.28$, both “both types” and “long-term only” equilibrium exists.

Second, we can investigate how profitability is affected by the variance of the shock (see Figure 4). The profit level is constant in the first regime, increasing in $k$ in the second regime, and decreasing in $k$ in the third regime. For small values of $k$, the “short-term only” regime is more profitable than the “both types of contracts” regime. However, for larger values of $k$ we get the opposite result. Finally, the “long-term
only” regime yields lower profits than the other two regimes.

Finally, we analyze consumer welfare. The question that we are interested in, is whether or not consumers benefit from the presence of long-term contracts. The answer is ambiguous, it depends on the shock-distribution that is, in this example, it depends on \( k \).

For large values of \( k \) \((k > 1.25)\), the equilibrium regime is the “long-term only” regime. Moreover, at these values of \( k \), this is what all consumers prefer the most. The low prices offset the extra transportation costs due to lock-in.

For small values of \( k \) \((k < 1)\), we always have the “short-term only” equilibrium. However, this is not the consumers’ most preferred regime. They all would, in fact, prefer if firms offered both types of contracts. Those, who buy short-term contracts would pay only \( t + kt \), compared to the payment of \( 2t \) under the “short-term only” regime. The long-term contractors would pay even less (since long-term contracts are sold at a discount) but they would bear some extra transportation costs. However, by a revealed preference argument we can show that they too, are better off, since they always have the option of buying short-term contracts.

Finally, for intermediate values of \( k \), \((1 < k < 1.25)\) the answer is not as clear-cut. It turns out that the costs and benefits of the presence of long-term contracts are not equally distributed. In general, some consumers would like to have a “short-term only” regime, whereas other consumers would like to see a “both types of contracts” equilibrium. In particular, those who buy short-term contracts, would pay less under the “short-term only” regime, thus they would prefer that regime. On the other hand, long-term contractors would pay less than \( 2t \), on total, in the “both types” regime (at least for smaller \( k \)'s). Although they bear extra transportation costs, this might not offset the benefit of lower prices. So, at least customers whose type is near to one of the extremes (these are the ones who bear the least extra transportation costs) would prefer the “both type of contracts” regime.

4 Extension: Endogenous switching costs

In the previous sections we have assumed that a long-term contract can not be breached. We made this simplification in order to keep the algebra within hands. However, it would be more realistic to assume that even long-term contracts can be cancelled, at the cost of paying a penalty fee\(^{13} \). So, more specifically, consider the following modification. In the first period, together with \( l_i \) and \( f_i \), firms set a cancellation fee, \( c_i \). In the second period, customers who bought long-term contracts would have the possibility to break it, pay the cancellation fee, and switch to the other firm. Of course, switchers can only take the short-term contract of the other firm.

\(^{13}\)If a contract could be breached without having to pay a penalty, we would not call it a long-term contract.
In this setting, we have two additional second-period cutoffs, \( L_A \) and \( L_B \), corresponding to customers who bought in the first period \( LT_A \) and \( LT_B \), respectively. For example, the choice rule of the first group of customers would be: continue with \( LT_A \) if \( 0 \leq x_2 \leq L_A \), otherwise break the contract, and buy \( ST_B \). It is straightforward to calculate these cutoffs.

\[
\begin{align*}
L_A &= \frac{1}{2} + \frac{-l_A + s_B + c_A}{2t} \\
L_B &= \frac{1}{2} - \frac{-l_B + s_A + c_B}{2t}
\end{align*}
\]

In equilibrium, if all four types of contracts are sold, it must be true that

\[ f_i \leq l_i + c_i \]

Otherwise, no customer would buy the contract \( ST_i \), because it would be cheaper to buy \( LT_i \) and breach it eventually, if the taste shock turns out to be sufficiently unfavorable. This result, together with the earlier stated \( 2l_i \leq f_i + s_i \), implies that \( l_i \leq c_i + s_i \). In a symmetric equilibrium, where \( s_A = s_B \), this further implies that

\[ L_B \leq \frac{1}{2} \leq L_A \]

This result is intuitive, it tells that those who bought long-term contracts switch firms with smaller probability than those who bought short-term contracts. For example, a customer at \( x_1 \in [0, 1/2] \) would switch with probability \( 1 - F(L_A | x_1) \) had he bought \( LT_A \), whereas he would switch with probability \( 1 - F(1/2 | x_1) \) had he bought \( ST_A \).\(^{14}\)

The solution concept to this game is exactly the same as to the basic model, except that firms now optimize over three prices in the first period, instead of two. We would have the same three types of symmetric equilibria. The “short-term contracts only” equilibrium would obviously be the same. Also, it is possible to show that in the “both types of contracts” equilibria, Proposition 2, that is, \( f_A = f_B = t \) still holds.

5 Conclusion

We conclude with some remarks about the switching cost framework that we have presented in the previous section. Switching costs, together with long-term contracts have been generally considered as anti-competitive. They lock-in customers and, as a consequence, firms can raise prices, thereby reducing

\(^{14}\) And, of course, \( 1 - F(L_A | x_1) < 1 - F(1/2 | x_1) \).
consumer welfare. However, most existing models consider switching costs as being some inherent (often non-monetary) costs that arise when the buyer chooses a different product (or supplier). Klemperer (1995) cites several reasons why a continued relationship with a certain product (or supplier) can be more convenient than switching. These models usually conclude that long-term contracts and switching costs hurt consumers and benefit firms.

There are interesting real-world examples, however, when there are no “inherent” switching costs, but rather suppliers “create” them. An example is when a retail store rewards buyers that continue buying his products. Another example is a service provider (cellular phone company) that offers long-term contracts, together with large cancellation fees. These examples are instances of endogenous switching costs, ones that are created artificially. It is not clear at all a priori whether these “competitively created” switching costs would also hurt consumers and benefit firms as in the models of exogenous switching costs. In fact, if these contracts have close substitutes in the form of repeated short-term contracts, then long-term contracts must be priced at a discount. This discount might offset the inconvenience of being locked-in, hence might be, in the end, beneficial to consumers.

References


A Appendix: Proofs

To facilitate reading, we summarize the notations used in the paper:

Contracts
\( LT_A, LT_B \) long-term contracts (switching between firms is NOT allowed)
\( ST_A, ST_B \) short-term contracts (switching between firms is allowed)

Prices
\( l_A, l_B \) long-term contract (per period price)
\( f_A, f_B \) 1\(^{st}\) period short-term contract
\( s_A, s_B \) 2\(^{nd}\) period short-term contract

Profits
\( \Pi_A, \Pi_B \) total profits
\( \Pi^2_A, \Pi^2_B \) profits from second-period short-term contracts

Cutoff values
1\(^{st}\) period
\( A \) customers with \( 0 \leq x_1 \leq A \) buy \( LT_A \)
\( B \) customers with \( B \leq x_1 \leq 1 \) buy \( LT_B \)
in case where both \( ST_A \) and \( ST_B \) are sold,
\( C \) customers with \( A \leq x_1 \leq C \) buy \( ST_A \), customers with \( C \leq x_1 \leq B \) buy \( ST_B \)

2\(^{nd}\) period
\( S \) among those, who have bought short-term contracts in the 1\(^{st}\) period,
customers with \( 0 \leq x_2 \leq S \) buy \( ST_A \), customers with \( S \leq x_2 \leq 1 \) buy \( ST_B \)

Customers’ types
\( x_1 \) customer’s location in the 1\(^{st}\) period
\( x_2 \) customer’s location in the 2\(^{nd}\) period

Parameters and distributions
\( f(x_2 | x_1) \) conditional pdf of the customer’s 2\(^{nd}\)-period type distribution
\( F(x_2 | x_1) \) conditional cdf of the customer’s 2\(^{nd}\)-period type distribution
\( k \) length of the support of the above distribution
\( t \) transportation cost of a unit distance
\( \Delta T(contract_1 ; contract_2) \) difference in expected “transportation” costs between \( contract_1 \) and \( contract_2 \)
**Proposition 1** The system (9) has a symmetric interior solution, provided that the following conditions are met.

i) \[ 1 + 4 \int_0^1 (1 - \Phi (\varepsilon)) \, d\varepsilon > -\frac{1}{\varphi (0)} \]

ii) Either

\[ h > 2 \frac{-8H^3 + 16H^2 - 4H - 1}{7 - 2H} \]

or

\[ h < H (-4H^2 + 8H - 3) \]

where \( h = \varphi (1/2), \ H = \Phi (1/2) \)

**Proof.** Since we are focusing on a symmetric solution, we will not use subscripts to denote firms. The system (9) is continuous in all variables. We show that the above conditions guarantee that the value of the function has different signs at the two extremes (at “small enough” \( l \) and at “large enough” \( l \)). Then, by the intermediate-value theorem, there is at least one interior root.

First of all, let us rewrite \( \Omega \), using the fact that the solution must be symmetric, that is \( S = 1/2, \ B = 1 - A, \ s_A = s_B = s \). From these and our distributional assumptions it follows also that \( F (S | B) = 1 - F (S | A), \ f (S | A) = f (S | B) \), and

\[
\int_{-1/2+A}^{1/2-A} \varphi (\varepsilon) \, d\varepsilon = 2\varphi (1/2 - A) - 1
\]

In addition,

\[
\int_{-1/2+A}^{1/2-A} \frac{\partial}{\partial \varepsilon} \varphi (\varepsilon) \, d\varepsilon = \varphi (1/2 - A) - \varphi (-1/2 + A) = 0
\]

Therefore, \( \Omega \) becomes

\[
\Omega = \begin{bmatrix}
2t (1 - \Phi (1/2 - A)) & 0 & -\Phi (1/2 - A) & -1 + \Phi (1/2 - A) \\
0 & -2t (1 - \Phi (1/2 - A)) & -1 + \Phi (1/2 - A) & -\Phi (1/2 - A) \\
-\Phi (1/2 - A) + \frac{\Phi (1/2 - A)}{2t} & 1 - \Phi (1/2 - A) + \frac{\Phi (1/2 - A)}{2t} & -\frac{2\Phi (1/2 - A) - 1}{t} & \frac{2\Phi (1/2 - A) - 1}{2t} \\
-1 + \Phi (1/2 - A) + \frac{\Phi (1/2 - A)}{2t} & \Phi (1/2 - A) & 2\Phi (1/2 - A) - 1 & -2\Phi (1/2 - A) - 1 \\
+\frac{\Phi (1/2 - A)}{2t} & -\frac{\Phi (1/2 - A)}{2t} & 2\Phi (1/2 - A) - 1 & \frac{2\Phi (1/2 - A) - 1}{t}
\end{bmatrix}
\]

27
Case 1. $A = B = 1/2$

Here, we consider a “small enough” $l$, so that every customer buys long-term contracts. Let us first find
the second-period prices. From (7) we get

$$s = 2l \frac{\int_A^{1-A} F (1/2|x_1) \, dx_1}{\int_A^{1-A} f (1/2|x_1) \, dx_1}$$

However, this expression has no meaning at $A = 1/2$, because both the numerator and the denominator
are equal to zero. Nevertheless, the limit exists:

$$\lim_{A \to 1/2} s = \frac{t}{\varphi (0)} \quad \text{(A.1)}$$

In general, $\{A, B, s_A, s_B\}$ are jointly determined by the system $\{g^A = 0, g^B = 0, \frac{\partial \Pi_A}{\partial s_A} = 0, \frac{\partial \Pi_B}{\partial s_B} = 0\}$, by
equations (4),(5),(7) and (8). However, in the current case, since $\Pi_A = \Pi_B = 0$, thus $\frac{\partial \Pi_A}{\partial s_A}$ and $\frac{\partial \Pi_B}{\partial s_B}$ are
identically equal to zero, so $A$ and $B$ are jointly determined by the system $\{g^A = 0, g^B = 0\}$.\footnote{Alternatively, one could try to evaluate $\Omega$ at $A = 1/2$. The last two rows of this matrix would have only 0's, so it would
not be invertible.} With this, the first equation in (9) simplifies to:

$$\frac{d\Pi_A}{dl_A} = \frac{\partial \Pi_A}{\partial l_A} + \left[ \frac{\partial \Pi_A}{\partial s_A} \quad \frac{\partial \Pi_A}{\partial B} \right] = 1 - \left[ t - \frac{s}{2} \right]^{-1} \left[ \begin{array}{cc} t & 0 \\ 0 & -t \end{array} \right] \left[ 2 \right]$$

From the equation for cutoff $A$, (4), we can express $2l$:

$$2l = t + s - 2l \int_{1/2}^{1/2+k/2} (1 - F (x_2|1/2)) \, dx_2 = t + s - 2l \int_0^{k/2} (1 - \Phi (\varepsilon)) \, d\varepsilon \quad \text{(A.2)}$$

Now, substituting (A.1) and (A.2) into the first-order condition, we get

$$\frac{d\Pi_A}{dl_A} = 1 - 2 \left( \frac{1}{2\varphi (0)} - 2 \int_0^{k/2} (1 - \Phi (\varepsilon)) \, d\varepsilon \right) = 1 - \frac{1}{\varphi (0)} + 4 \int_0^{k/2} (1 - \Phi (\varepsilon)) \, d\varepsilon$$

By assumption, this expression is positive, meaning that profits would increase, if firms raised long-term
prices.

Case 2. $A = 0, B = 1$
Now, we consider “large” $l$, so that all customers buy short-term contracts only. Let us use the following shorthands: $h = \varphi(1/2), H = \Phi(1/2)$. The short-term price will be equal to

$$s = 2t \frac{\int_0^1 F(1/2 | x_1) \, dx_1}{\int_0^1 f(1/2 | x_1) \, dx_1} = \frac{t}{2H - 1}$$

Substituting $s$ into $\Omega$, then $\Omega$ into (9), we get

$$\frac{d\Pi_A}{dl_A} = \frac{(1 - H) (-16H^3 + 32H^2 - 8H - 7h + 2hH - 2)}{2(2H - 1)(4H^4 - 16H^3 + 19H^2 - 6H + hH - 2h)}$$

This expression is negative if and only if $(-16H^3 + 32H^2 - 8H - 7h + 2hH - 2)$ and $(4H^4 - 16H^3 + 19H^2 - 6H + hH - 2h)$ have the same signs. It is straightforward to verify that the conditions denoted by $ii)$ in this Proposition guarantee that this holds. Therefore, at large $l$, $\frac{d\Pi_A}{dl_A}$ is negative, meaning that firms would increase profits by decreasing $l$. ■

**Proposition 3** If $0 \leq k \leq 1$ then there exists a symmetric equilibrium, in which firms offer only short-term contracts. The equilibrium prices are the following:

$$f_A = f_B = t$$

$$s_A = s_B = t$$

In addition, these bounds are sharp, i.e. when $1 < k \leq 2$, then the above strategies do not constitute an equilibrium.

**Proof.** (See Prop1_3.tex) First of all, let us verify that these are indeed the equilibrium prices when firms offer only short-term contracts. If only short-term contracts are available then no customer is committed in the second period, so the game as a whole is identical to a repetition of two one-shot games. We show that regardless of the value of $k$, the equilibrium solution is the same. Intuitively, this should be clear, because any identically distributed shock would result in an invariant distribution (i.e. the first-period distribution is identical to the second-period distribution). Formally, consider first $k \leq 1$. Profits are the following:

$$\Pi_A^2 = s_A \int_0^1 F(S | x_1) \, dx_1 = s_A \left( \int_0^{S - \frac{1}{2}} dx_1 + \int_{S - \frac{1}{2}}^{S + \frac{1}{2}} \left( \frac{S - x_1}{k} + \frac{1}{2} \right) \, dx_1 \right) = s_A S$$

$$\Pi_B^2 = s_B \int_0^1 1 - F(S | x_1) \, dx_1 = s_B \left( \int_{S - \frac{1}{2}}^{S + \frac{1}{2}} \left( 1 - \frac{S - x_1}{k} - \frac{1}{2} \right) \, dx_1 + \int_{S + \frac{1}{2}}^1 dx_1 \right) = s_B (1 - S)$$
Now, consider \( k > 1 \). Profits are the following:

\[
\Pi_A^2 = s_A \int_0^1 F(S \mid x_1) \, dx_1 - s_A \int_0^{\frac{k}{k-S}} F(-S \mid x_1) \, dx_1 + s_A \int_{2-S-\frac{k}{k}}^1 (1 - F(2 - S \mid x_1)) \, dx_1 = \\
= s_A \left( \int_0^1 \left( \frac{S - x_1}{k} + \frac{1}{2} \right) \, dx_1 - \int_0^{\frac{k}{k-S}} \left( \frac{-S - x_1}{k} + \frac{1}{2} \right) \, dx_1 + \int_{2-S-\frac{k}{k}}^1 \left( 1 - \frac{2 - S - x_1}{k} - \frac{1}{2} \right) \, dx_1 \right) \\
= s_A S
\]

\[
\Pi_B^2 = s_B \int_0^1 (1 - F(S \mid x_1)) \, dx_1 + s_B \int_0^{\frac{k}{k-S}} F(-S \mid x_1) \, dx_1 - s_B \int_{2-S-\frac{k}{k}}^1 (1 - F(2 - S \mid x_1)) \, dx_1 = \\
= s_B \left( \int_0^1 \left( 1 - \frac{S - x_1}{k} - \frac{1}{2} \right) \, dx_1 + \int_0^{\frac{k}{k-S}} \left( \frac{-S - x_1}{k} + \frac{1}{2} \right) \, dx_1 - \int_{2-S-\frac{k}{k}}^1 \left( 1 - \frac{2 - S - x_1}{k} - \frac{1}{2} \right) \, dx_1 \right) \\
= s_B (1 - S)
\]

So, either way, the profits are equal to the same expression. It is easy to verify that the unique solution is \( s_A = s_B = t \). The determination of the first-period prices is a standard exercise. We get \( f_A = f_B = t \).

The above strategies, are indeed an equilibrium if and only if \( k \leq 1 \). We show that knowing that the other firm plays the above strategy, either firm would find it profitable to deviate if and only if \( k > 1 \). To start, assume that firm \( B \) offers only short-term contracts and charges \( f_B = t \). Then suppose that \( A \) offers both types of contracts at prices \( l_A \) and \( f_A \). It is straightforward to prove that the optimal \( f_A \) is \( t \). Customers will buy both short-term and long-term contracts from firm \( A \) if and only if \( A < \frac{1}{2} \) and \( S \leq A + \frac{k}{2} \). (The second requirement is the result of Lemma 1.) Otherwise \( A \) will be able to sell only long-term contracts.

The rest of the proof consists of a number of different cases.

**Case 1.** \( S \notin [A - k/2, A + k/2] \) (See subcase1.mws)

According to Lemma 1, it is always the case that \( A - \frac{k}{2} \leq S \), therefore this case is equivalent to \( A + \frac{k}{2} < S \). Hence, \( A \) sells only long-term contracts. We solve for the second-period equilibrium outcome in Lemma 2. It is \( s_A = (1 - \frac{k}{2} A) t \), \( s_B = (1 - \frac{k}{2} A) t \), \( S = \frac{1}{2} + \frac{1}{3} A \), \( \Pi_A^2 = \left( \frac{1}{2} - \frac{1}{2} A + \frac{5}{6} A^2 \right) t \). Furthermore, the marginal customer (the one located at \( A \)) will buy from \( A \) in the second period with certainty. Thus, the equation that determines the cutoff is the following: \( 2l_A + 2At = f_B + s_A + (1 - A)t + At \). Rearranging this yields: \( l_A = (1 - A)t + \frac{1}{2}s_A \). So, after simplifying, the profit of firm \( A \) is:

\[
\Pi_A = 2Al_A + \Pi_A^2 = \left( -\frac{22}{9} A^2 + \frac{5}{3} A + \frac{1}{2} \right) t < t \quad \forall A \in [0, 1]
\]
Case 2. $S \in [A - k/2, A + k/2]$

We solve for the second-period equilibrium outcome in Lemma 3, see equations (A.4), (A.5) and (A.6). Rearranging (A.6), it is easy to show that $S \leq A + \frac{k}{2}$ is equivalent to the condition $A \geq \frac{3}{4}(1 - k)$. We will use this condition several times later on. We will consider three subcases here; in the first two $A \geq \frac{1}{2}$ (so $A$ sells only long-term contracts), in the third $A < \frac{1}{2}$ (so $A$ sells both types of contracts).

Subcase 2.1. $A \geq \frac{1}{2}$ and $0 \leq k \leq \frac{1}{3}$ (See subcase21.mws)

$A$ sells only long-term contracts. First, we show that $S \geq \frac{1}{2}$. Whenever $k \leq \frac{1}{3}$, we have $S \geq A - \frac{k}{2} \geq \frac{3}{4}(1 - k) - \frac{k}{2} = \frac{1}{4} - \frac{3k}{4} \geq \frac{1}{3}$. If $\frac{1}{3} \leq k \leq \frac{1}{2}$, then we use equation (A.6) of Lemma 3. After a simple manipulation it yields the following expression: $8 \left( S - \frac{1}{2} \right) = 3(2A - k - 1) + \sqrt{(1 + k - 2A)^2 + 32k(1 - A)}$. This term is negative only if $A < \frac{1 + k}{2}$. Therefore $8 \left( S - \frac{1}{2} \right) \geq -3k + \sqrt{32k(1 - \frac{1 + k}{2})} = -3k + 4\sqrt{k(1 - k)} \geq 0$ if $k \leq \frac{16}{25}$.

Second, $S \geq \frac{1}{2}$ implies that in the second period firm $B$ charges more than firm $A$, because (using (A.4) and (A.19) in Lemma 3): $s_B - s_A = t (3S + \frac{k}{2} - A - 1) - t (S + \frac{k}{2} - A) = t(2S - 1) \geq 0$.

Third, we establish a loose upper bound on $l_A$. Since $A$ sells only long term contracts, $A$ is the location of the customer who is indifferent between $LT_A$ and $ST_B$. Now consider the following inequality, which imposes an upper bound on $l_A$. $2l_A + At \leq f_B + s_B + (1 - A)t$. This inequality is not sharp, for two reasons. First, on the right-hand side, the customer’s expected second-period payment is generally less than $s_B$, because in expectation, he pays some weighted average of $s_A$ and $s_B$. Second, we did not take into account the expected second-period transportation costs, which would be greater on the left-hand side, since that is a long-term contract. Rearranging this inequality and substituting $f_B = t$ and $s_B$ yields: $l_A \leq \left( -\frac{3}{8}A + \frac{11}{16} - \frac{5}{16}k + \frac{3}{16}\sqrt{(1 + k - 2A)^2 + 32k(1 - A)} \right)t$.

Finally, we establish an upper bound on the second-period profit as well: since $S \leq A + \frac{k}{2}$, $\Pi_A^2 = \frac{(\frac{3}{2} - A - A)^2}{2k} t \leq \frac{1}{2}k^2 t$.

Hence, putting all these together, we get

$$\Pi_A = 2l_A + \Pi_A^2 \leq \left( -\frac{3}{4}A^2 + \frac{11}{8}A - \frac{5}{8}Ak + \frac{3}{8}A\sqrt{(1 + k - 2A)^2 + 32k(1 - A)} + \frac{1}{2}k^2 \right)t \leq t$$

(We need to prove this yet.)

Subcase 2.2. $A \geq \frac{1}{2}$ and $\frac{1}{3} \leq k < 1$ (See subcase22.mws)

$A$ sells only long-term contracts. Unlike in Subcase 2.1., here we establish a sharp upper bound on $l_A$. The expected payments of contracts $LT_A$ and $ST_B$, and the difference in transportation costs between the two contracts are the following:
\[
\begin{align*}
P_{LT_A} &= 2l_A \\
P_{ST_B} &= f_B + s_A \left( \frac{S - A}{k} + \frac{1}{2} \right) + s_B \left( \frac{-S + A}{k} + \frac{1}{2} \right) \\
\Delta T(LT_A; ST_B) &= (2A - 1) t + \frac{t}{k} \left( A^2 - A - S^2 + S + Ak + \frac{1}{4} k^2 - \frac{1}{2} k \right)
\end{align*}
\]

The cutoff is determined from the equality \( P_{LT_A} - P_{ST_B} + \Delta T(LT_A; ST_B) = 0 \). We refer to Lemma 3, which gives us the equilibrium second-period prices. The profit is \( \Pi_A = 2Al_A + \frac{(S + \frac{1}{2} - A)^2}{2k} \). Substitute everything into the profit function and express it in terms of \( A \) to get

\[
\Pi_A = -\frac{5}{32} A^3 t + \frac{20\delta - 484k + 44}{256k} A^2 t + \frac{(76k - 12) \delta - 198k^2 + 380k - 14}{256k} A t + \frac{(k^2 + 10k + 1) \delta + k^3 + 27k^2 + 27k + 1}{256k}
\]

where \( \delta = \sqrt{(1 + k - 2A)^2 + 32b(1 - A)} \).

Provided that \( A \geq \frac{1}{2} \), \( \Pi_A \) is increasing in \( \delta \) (the derivative is \( \frac{20A^2 + 76A + 12A + k^2 + 10k + 1}{256k} \)). Therefore, to set an upper bound on \( \Pi_A \) substitute for \( \delta \) its highest possible value. It is easy to verify that provided \( k \geq \frac{1}{3} \), \( \delta \) is decreasing in \( A \). Therefore \( \delta \) is maximized at \( A = \frac{3}{4} (1 - k) \), because this is the smallest admissible \( A \). Hence \( \delta \leq \frac{11}{12} \). As a consequence, we can bound the profit from above as:

\[
\Pi_A \leq -\frac{5}{32} A^3 t + \frac{748k + 108}{512k} A^2 t + \frac{440k^2 + 704k - 40}{512k} A t + \frac{13k^3 + 165k^2 + 75k + 3}{512k}
\]

The right hand side is a 3rd-order polynomial in \( A \), with negative coefficient on the highest order term. Therefore its maximizer is either the smallest possible \( A \), or the larger root of the first derivative, if this latter is not greater than 1 (otherwise we would have to check \( A = 1 \)). Check first \( A = \frac{3}{4} (1 - k) \). In this case the profit simplifies to \( \Pi_A = -\frac{11k^2 + 12k + 3}{8} t \). It is easy to show that this is always less than \( t \). Next, we check the larger root of the first derivative, which is \( A = \frac{9}{20} - \frac{187}{600} k + \frac{1}{60} \sqrt{41569k^2 + 462k + 129} \). This lies indeed in the interval \([1/2, 1]\), and it is straightforward to verify that \( \Pi_A \) is always less than \( t \) at this \( A \) as well.

**Subcase 2.3.** \( A < \frac{1}{2} \) (See subcase23.mws, subcase23.tex, subcase23.m)

Firm \( A \) sells both types of contracts. The cutoff level \( A \) is the location of the marginal customer, who is indifferent between \( LT_A \) and \( ST_A \). It is easy to see that this case only applies if \( k \geq \frac{1}{3} \). Otherwise we would have \( A \geq \frac{3}{4} (1 - k) > \frac{1}{2} \), a contradiction. The expected payments of contracts \( LT_A \) and \( ST_A \), and the difference in transportation costs between the two contracts are the following:
\[ P_{LT\alpha} = 2l_A \]
\[ P_{ST\alpha} = f_A + s_A \left( \frac{S - A}{k} + \frac{1}{2} \right) + s_B \left( \frac{-S + A}{k} + \frac{1}{2} \right) \]
\[ \Delta T (LT\alpha; ST\alpha) = \frac{t}{k} \left( A^2 - A - S^2 + S + Ak + \frac{1}{4}k^2 - \frac{1}{2}k \right) \]

The cutoff is determined from the equality \( P_{LT\alpha} - P_{ST\alpha} + \Delta T (LT\alpha; ST\alpha) = 0 \). Again, we refer to Lemma 3, which gives us the equilibrium second-period prices. The profit is \( \Pi_A = 2A l_A + t \left( \frac{S}{2} - A \right) + s_A \frac{(S + \frac{S}{2} - A)^2}{2k} \).

When we substitute everything into the profit function and express it in terms of \( A \) we get
\[ \Pi_A = -\frac{5}{32} A^3 t + \frac{20\delta + 28k + 44}{256k} A^2 t - \frac{(-76k + 12) \delta + 198k^2 + 132k + 14}{256k} At \]
\[ + \frac{(k^2 + 10k + 1) \delta + k^3 + 27k^2 + 155k + 1}{256k} \]

where \( \delta = \sqrt{(1 + k - 2A)^2 + 32k(1 - A)} \)

For all \( k \in [1/3, 1] \), \( A \in [0, 1/2] \), this is a strictly decreasing function of \( A \). Therefore, the maximizer is \( A = 0 \). Then \( \Pi_A \) simplifies to \( \Pi_A = \frac{(k^2 + 10k + 1)\delta + k^3 + 27k^2 + 155k + 1}{256k} t \). It is straightforward to verify (by taking the first derivative) that this expression is strictly increasing in \( k \) for all positive \( k \). At \( k = 1 \) it is exactly equal to \( t \). Therefore, for all \( k \in [1/3, 1] \), the profit is going to be strictly less than \( t \).

On the other hand, this exercise shows us that it is possible to make a profitable deviation if \( k \geq 1 \). Even \( A = 0 \) would yield a profit level greater than \( t \). However, this is not the optimal strategy, firm \( A \) can do even better. It turns out that when \( k > 1 \), then the maximizer of (A.3) is no longer 0, but some \( A \in (0, 1/2) \).

The graph below shows the optimal strategy.

**Lemma 1** In an equilibrium, where customers buy both long-term and short-term contracts from \( A \), it must be the case that \( A + \frac{k}{2} \geq S \). Similarly, if customers buy both long-term and short-term contracts from \( B \), it must be that \( B - \frac{k}{2} \leq S \).

**Proof.** This statement is almost obvious. Customers at or to the left of \( S - \frac{k}{2} \) will never get a second period draw above \( S \). Therefore, these customers are certain that they will buy from firm \( A \) in both periods. Hence, transportation costs are the same, regardless of what type of contract they buy from \( A \). Thus, all of them make their decisions by comparing the total price of the long-term contract to the total price of two consecutive short-term contracts, that is \( 2l_A \) to \( f_A + s_A \). All of them will simply choose the
cheaper of these two. Hence the cutoff level cannot be in the range \([0, S - k/2]\). The proof of the other part of the statement goes along the same lines. \(\blacksquare\)

**Lemma 2** Suppose \(A\) offers long-term (and possibly short-term) contracts, while \(B\) offers only short-term contracts (therefore \(B = 1\)). If the second-period equilibrium satisfies \(A + \frac{k}{2} < S\) then the short-term prices and the cutoff will be the following:

\[
\begin{align*}
s_A &= \left(1 - \frac{4}{3}A\right) t \\
s_B &= \left(1 - \frac{2}{3}A\right) t \\
S &= \frac{1}{2} + \frac{1}{3}A
\end{align*}
\]

**Proof.** It is straightforward. The second period cutoff is \(S = \frac{1}{2} - \frac{s_A - s_B}{2t}\). Profits from the second-period spot contracts are the following:
Figure 6: Equilibrium profit of A when A offers $LT_A$ and $ST_A$, while B offers only $ST_B$.

\[
\Pi_A^2 = s_A \left( \int_A^{S - \frac{k}{2}} dx_1 + \int_{S - \frac{k}{2}}^{S + \frac{k}{2}} \left( \frac{S - x_1}{k} + \frac{1}{2} \right) dx_1 \right) = s_A (S - A)
\]

\[
\Pi_B^2 = s_B \left( \int_{S - \frac{k}{2}}^{S + \frac{k}{2}} \left( -\frac{S + x_1}{k} + \frac{1}{2} \right) dx_1 + \int_{S + \frac{k}{2}}^{1} dx_1 \right) = s_B (1 - S)
\]

It is easy to verify that these functions are globally concave. Solving the first-order conditions yield the above prices. ■

**Lemma 3** Suppose A offers both short-term and long-term contracts, while B offers only short-term contracts (therefore $B = 1$). If the second-period equilibrium satisfies $S \leq A + \frac{k}{2}$ then the short-term prices and the cutoff will be the following:

\[
s_A = \left( S + \frac{k}{2} - A \right) t \tag{A.4}
\]

\[
s_B = \left( 3S + \frac{k}{2} - A - 1 \right) t \tag{A.5}
\]

\[
S = \frac{1}{8} \left( 6A - 3k + 1 + \sqrt{(1 + k - 2A)^2 + 32k(1 - A)} \right) \tag{A.6}
\]
Also, with these prices, \( A \leq S + \frac{k}{2} \). Also, in equilibrium, the profit of firm \( A \) from second-period spot contracts is \( \frac{(S + \frac{k}{2} - A)^3}{2k} \).

**Proof.** (See lemma3.mws) The second period cutoff is \( S = \frac{k}{2} - \frac{4a - 6a}{2A} \). Assume that \( A \leq S + \frac{k}{2} \). We will check at the end whether the solution satisfies this condition.

First, we calculate the profits from the second-period spot contracts. Consider two cases: \( S + \frac{k}{2} \leq 1 \), and \( S + \frac{k}{2} \geq 1 \). In the former case:

\[
\Pi_A^2 = s_A \int_A^{S - \frac{k}{2}} F(S | x_1) \, dx_1 = s_A \int_A^{S - \frac{k}{2}} \left( \frac{S - x_1}{k} + \frac{1}{2} \right) \, dx_1 = s_A \frac{(S + \frac{k}{2} - A)^2}{2k}
\]

\[
\Pi_B^2 = s_B \int_A^{S - \frac{k}{2}} (1 - F(S | x_1)) \, dx_1 + s_B \left( 1 - S - \frac{k}{2} \right) = s_B \frac{(S + \frac{k}{2} - A)^2 + 2k(1 - A)}{2k}
\]

In the latter case:

\[
\Pi_A^2 = s_A \left( \int_A^{1} F(S | x_1) \, dx_1 + \int_{2 - S - \frac{k}{2}}^{1} (1 - F(2 - S | x_1)) \, dx_1 \right) = s_A \frac{(S + \frac{k}{2} - A)^2}{2k}
\]

\[
\Pi_B^2 = s_B \left( \int_A^{1} (1 - F(S | x_1)) \, dx_1 - \int_{2 - S - \frac{k}{2}}^{1} (1 - F(2 - S | x_1)) \, dx_1 \right) = s_B \frac{(S + \frac{k}{2} - A)^2 + 2k(1 - A)}{2k}
\]

We get the same expressions for profits in both cases. After substituting for \( S \), we can notice that \( \Pi_A^2 \) is a cubic polynomial in \( s_A \), with positive coefficient on the highest order term. Similarly, \( \Pi_B^2 \) is a cubic polynomial in \( s_B \), but with negative coefficient on the highest order term. We require nonnegativity of prices as well as the assumed constraint, \( S \leq A + \frac{k}{2} \). Given that we have cubic polynomials, if the solution is interior, then \( s_A \) is the smaller root of its first-order condition, whereas \( s_B \) is the larger root of its first-order condition. The two roots of the two first-order conditions are the following:

\[
\left\{ \frac{1}{3} \left( t + kt + s_B - 2At \right), (t + kt + s_B - 2At) \right\}
\]

\[
\left\{ \frac{2}{3} \left( s_A + 2At - t - kt \right) \pm \frac{1}{3} \sqrt{s_A \left( s_A + 4At - 2t - 2kt \right) + t^2 \left( (1 + k - 2A)^2 + 24k(1 - A) \right)} \right\}
\]
We argue that the solution must be interior. If the constrained maximizer of $\Pi_A^2$ is not the smaller root (interior solution) then it must be greater than the larger root. Similarly, if the constrained maximizer of $\Pi_B^2$ is not the larger root (interior solution) then it must be less than the smaller root. Now, if $s_A > t + kt + s_B - 2At$, then $A + \frac{k}{2} < S$, which violates our assumption in the statement of this lemma. Hence, $s_A = \frac{1}{4}(t + kt + s_B - 2At)$. Now, examine what would happen if $s_B$ was less than or equal to the smaller root. It is easy to see that $s_A = \frac{1}{4}(t + kt + s_B - 2At)$ together with $s_B < \frac{2}{3}(s_A + 2At - t - kt)$ implies: $s_A < \frac{1+k-2A}{2}$, $s_B < \frac{-4-4k+8A}{3}$, therefore at least one of the prices would be negative, which is not admissible. Therefore both maximization problem must have interior solution. If we solve this system, we get the prices as stated above. Finally, let’s check whether our starting assumption is met at the solution. Indeed,

$$S + \frac{k}{2} - \frac{3}{4} = \frac{1}{8}\left(k + 1 + \sqrt{(1+k-2A)^2 + 32k(1-A)}\right) > \frac{1}{8}(k + 1 + |1+k-2A|) \geq \frac{1}{4}A$$

\[\blacksquare\]

**Proposition 4** If $0.81 \leq k \leq 1.28$ then there exists a symmetric equilibrium, in which firms offer both short-term and long-term contracts. The equilibrium prices and cutoffs are the following:

$$l_A = l_B = \frac{79k^2 + 164k - 4 + (2-k)d}{256k}$$

$$f_A = f_B = t$$

$$s_A = s_B = kt$$

$$A = \frac{6 - 7k + d}{16}$$

$$B = \frac{10 + 7k - d}{16}$$

where $d = \sqrt{(97k^2 - 68k + 4)}$

**Proof.** (See prop2.mws) Assume that $\frac{1}{2} \leq A + \frac{k}{7} \leq 1$, a condition to which we will come back at the end of the proof. We will check if our solution satisfies this condition. The expected payments and the difference in transportation costs between the two contracts are the following:

\[\text{These bounds are only approximate. The exact lower bound is the root of the following 8th order polynomial:} \]

$$38046501Z^8 - 80585340Z^7 - 9011246Z^6 + 63690268Z^5 - 1313407Z^4 - 11634304Z^3 - 2170424Z^2 - 127232Z - 2352 = 0.$$  

The exact upper bound is $\frac{1}{4} + \frac{\sqrt{17}}{4}$. 

37
\[ P_{LTA} = 2l_A \]
\[ P_{STA} = f_A + s_A F(S | A) + s_B (1 - F(S | A)) \]
\[ \Delta T(LTA; STA) = t \int_{S}^{A+B} (2x_2 - 1) f(x_2 | A) dx_2 \]

\( A \) can be determined from the equality \( P_{LTA} - P_{STA} + \Delta T(LTA; STA) = 0 \). After calculating the integrals, this reduces to the following condition:

\[ g^A(A, B, l_A) = 2l_A - f_A - s_A \left( \frac{S - A}{k} + \frac{1}{2} \right) - s_B \left( \frac{-S + A}{k} + \frac{1}{2} \right) \frac{t}{k} \left( A^2 - A - S^2 + S + Ak + \frac{1}{4}k^2 - \frac{1}{2}k \right) = 0 \]

In the same way, we can derive a similar equation for \( B \).

\[ g^B(A, B, l_A) = 2l_B - f_B - s_A \left( \frac{S - B}{k} + \frac{1}{2} \right) - s_B \left( \frac{-S + B}{k} + \frac{1}{2} \right) \frac{t}{k} \left( B^2 - B - S^2 + S - Bk + \frac{1}{4}k^2 + \frac{1}{2}k \right) = 0 \]

It is straightforward to show that in equilibrium, if firms offer both short-term and long-term contracts, the first-period spot prices are \( f_A = f_B = t \), and the cutoff \( C = \frac{1}{2} \).

Lemma 4 solves for the 2nd period equilibrium prices. We take the expression of the second-period profit from Lemma 4, to express total profit as:

\[ \Pi_A = 2Al_A + t \left( \frac{1}{2} - A \right) + s_A \frac{(B - A)(2S - B - A + k)}{2k} \] (A.7)

The natural way to proceed would be to solve the system formed by \( (g^A) \) and \( (g^B) \) for \( A \) and \( B \), substitute the resulting expressions as well as \( S \) and \( s_A \) (from Lemma 4) into (A.7), so that the profit becomes only a function of \( l_A \). The problem is that if we try to solve \( (g^A) \) and \( (g^B) \), the resulting \( A \) and \( B \) are roots of some 4th order polynomials, and generally they can not be expressed in closed form. Therefore, it is easier to proceed in a different way. Rather than trying to find the exact expressions for \( A \), and \( B \), it is enough to find their derivatives with respect to \( l_A \). Notice that, by the chain rule:

\[ \frac{d\Pi_A}{dl_A} = \frac{\partial \Pi_A}{\partial l_A} + \frac{\partial \Pi_A}{\partial A} \frac{dA}{dl_A} + \frac{\partial \Pi_A}{\partial B} \frac{dB}{dl_A} \] (A.8)

To find the partial derivatives of \( \Pi_A \), plug (A.19) and (A.21) into (A.7), then differentiate it:
\[
\begin{align*}
\frac{\partial \Pi_A}{\partial l_A} &= 2A \\
\frac{\partial \Pi_A}{\partial A} &= \frac{(B^2 - 3A^2 - 2AB + 4A + 12A k - 9k^2 - 24k - 1) t + 36l_A k}{18k} \\
\frac{\partial \Pi_A}{\partial B} &= \frac{(3B^2 - A^2 + 2AB - 4B - 12B k + 9k^2 + 6k + 1) t}{18k}
\end{align*}
\]

To find the derivatives of the cutoffs \(A\) and \(B\), with respect to \(l_A\), use the equilibrium 2nd period prices, (A.19), (A.20) and (A.21), to simplify \((g^A)\) and \((g^B)\):

\[
\begin{align*}
g^A (A, B, l_A) &= \frac{(16A^2 - 16AB + 4B^2 - 8A + 4B + 36A k - 27k^2 - 54k + 1) t + 72l_A k}{36k} = 0 \quad (A.9) \\
g^B (A, B, l_B) &= \frac{(16B^2 - 16AB + 4A^2 - 8B + 4A - 36B k - 27k^2 - 18k + 1) t + 72l_B k}{36k} = 0 (A.10)
\end{align*}
\]

Now, by totally differentiating the system formed by \((g^A)\) and \((g^B)\) we get:

\[
\begin{align*}
\frac{\partial g^A}{\partial A} dA + \frac{\partial g^A}{\partial B} dB + \frac{\partial g^A}{\partial l_A} dl_A &= 0 \\
\frac{\partial g^B}{\partial A} dA + \frac{\partial g^B}{\partial B} dB + \frac{\partial g^B}{\partial l_B} dl_B &= 0
\end{align*}
\]

Divide these equations by \(dl_A\) and arrange them in matrix form:

\[
\begin{bmatrix}
\frac{\partial g^A}{\partial A} & \frac{\partial g^A}{\partial B} \\
\frac{\partial g^B}{\partial A} & \frac{\partial g^B}{\partial B}
\end{bmatrix}
\begin{bmatrix}
\frac{dA}{dl_A} \\
\frac{dB}{dl_A}
\end{bmatrix}
= \begin{bmatrix}
-\frac{\partial g^A}{\partial l_A} dl_A \\
-\frac{\partial g^B}{\partial l_B} dl_B
\end{bmatrix}
\]

Now, since we are interested in the response of \(A\) and \(B\) to a change in \(l_A\), holding \(l_B\) unchanged, fix \(dl_B = 0\), and rearrange:

\[
\begin{bmatrix}
\frac{dA}{dl_A} \\
\frac{dB}{dl_A}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial g^A}{\partial A} & \frac{\partial g^A}{\partial B} \\
\frac{\partial g^B}{\partial A} & \frac{\partial g^B}{\partial B}
\end{bmatrix}
^{-1}
\begin{bmatrix}
\frac{\partial g^A}{\partial l_A} \\
0
\end{bmatrix}
\]

After we substitute into this latter expression our functional forms given in (A.9) and (A.10), we get
\[
\begin{align*}
\frac{dA}{dl_A} &= -\frac{6k(4A - 8B + 9k + 2)}{(8A^2 - 20AB + 8B^2 + 2A + 2B + 36Ak - 36Bk + 27k^2 - 1)t} \\
\frac{dB}{dl_A} &= \frac{6k(4B - 2A + 1)}{(8A^2 - 20AB + 8B^2 + 2A + 2B + 36Ak - 36Bk + 27k^2 - 1)t}
\end{align*}
\]

(A.11)

(A.12)

These partial derivatives can be plugged into (A.8), to get the first-order condition of firm A. The problem is that this yields an extremely complicated expression, that we can’t solve, at least not in closed form. However, since the whole problem is symmetric, we have strong reason to believe that there exists a symmetric solution. Therefore we will impose at this point the conditions \(l_A = l_B = l\), and \(B = 1 - A\). Further, after finding a symmetric solution, we will verify that it is indeed an equilibrium. Therefore the rest of the proof consists of two parts, the first deals with the necessary conditions, in the second we show that these are also sufficient.

**Necessity**

With the above mentioned restrictions the equations that determine the cutoffs, (A.9) and (A.10) simplify to:

\[
g^A (A, B, l_A) = g (A, l) = \frac{t \left( 4A^2 - 4A + 1 + 4Ak - 3k^2 - 6k + 8lk \right)}{4k} = 0
\]

(A.13)

Also, the first-order condition takes the following form:

\[
\frac{d\Pi_A}{dl_A} = \frac{(8A^3 - 8A^2 + 8A^2k + 2A + 8Ak + 4Ak^2 + 3k^3 + 7k^2 - 6k)t - 4k(3k + 4A - 2)l}{(2A + 3k - 1)(2A + k - 1)t} = 0
\]

(A.14)

The system formed by (A.13) and (A.14) has the following three roots:

\[
\left\{ \frac{3}{8} - \frac{7}{16}k - \frac{1}{16}\sqrt{97k^2 - 68k + 4}, \frac{1}{2} - \frac{k}{2}, \frac{3}{8} - \frac{7}{16}k + \frac{1}{16}\sqrt{97k^2 - 68k + 4} \right\}
\]

We can rule out two of these three roots. Lemma 1 states that \(A \geq S - \frac{k}{2}\). It is straightforward to prove that the first root is less than \(\frac{1}{2} - \frac{k}{2}\), and therefore it cannot be part of a symmetric equilibrium (where \(S = \frac{1}{2}\)). Lemma 5 proves that the second root can not be the equilibrium cutoff either. Hence the only possibility is the third (the largest) root. Indeed, this root satisfies all assumptions needed for to be a valid cutoff.

It is easy to verify that with \(A = \frac{3}{8} - \frac{7}{16}k + \frac{1}{16}\sqrt{97k^2 - 68k + 4}\), the assumption made at the start of the proof, \(\frac{1}{2} \leq A + \frac{k}{2} \leq 1\) is indeed true, provided that \(k \in [2/3, 1/4 + \sqrt{17]/4}\). Plugging \(A\) into (A.13), we get:

\[
l = \frac{79k^2 + 164k - 4 + (2 - k)\sqrt{97k^2 - 68k + 4}}{256k}t
\]

40
The equilibrium prices are \( l_A = l_B = l \).

**Sufficiency**

In this part, we verify that the candidate prices form indeed an equilibrium. Suppose, therefore, that firm \( B \) charges the equilibrium long-term prices: \( l_B = l \). We will use this assumption to rewrite the profit function of firm \( A \), and establish that the constrained maximizer is the same price. The problem, however, is that for some values of \( k \in [2/3, 1/4 + \sqrt{17}/4] \), the profit function is not globally concave. Additionally, we need to be concerned with corner solutions as well. If \( A \) sets \( l_A \) to be too low, then customers will choose only long-term contracts from firm \( A \). Similarly, if \( A \) sets \( l_A \) to be too high, customers will only choose short-term contracts from firm \( A \). In either of these cases, equation (A.9), that we have used to determine the cutoff level \( A \), is no longer valid. Equation (A.3), the profit function, is also not valid in these cases. Therefore, the profit, as a function of \( l_A \), will generally have two kinks, and we must compare the maxima of the three segments of the profit function.

First of all, let’s verify that the candidate equilibrium is at least a local maximum, by checking the second-order condition at the solution. We can apply the chain rule again:

\[
\frac{d^2 \Pi_A}{dl_A^2} = \frac{\partial}{\partial l_A} \left( \frac{\partial \Pi_A}{\partial A} \right) + \frac{\partial}{\partial A} \left( \frac{\partial \Pi_A}{\partial l_A} \right) \frac{dA}{dl_A} + \frac{\partial}{\partial B} \left( \frac{\partial \Pi_A}{\partial l_A} \right) \frac{dB}{dl_A}
\]  

(A.15)

When we differentiate (A.8) w.r.t. \( l_A \) and use (A.11) and (A.12) to obtain (A.15), the result is a very complicated expression. Numerical calculations have shown us that for some (low) \( k \)'s it could take positive sign, hence the function is generally not globally concave. However, if we substitute the candidate values of \( A, B \) and \( l_A \), then (A.15) “simplifies” to the following expression.

\[
\frac{d^2 \Pi_A}{dl_A^2} = -64 \frac{123k^2 - 106k + 8 + (19k - 4) \sqrt{97k^2 - 68k + 4}}{(17k - 2 + \sqrt{97k^2 - 68k + 4}) (57k^2 - 52k + 4 + (9k - 2) \sqrt{97k^2 - 68k + 4})} t
\]

It is straightforward to verify that this expression is negative for all \( k \in [2/3, 1/4 + 1/4\sqrt{17}] \).

This last result means that the profit function has a local maximum at \( l_A = l \). Now, to show that it is, in fact, a global maximum, we take a different approach. Consider again equations (A.9) and (A.10). Solve these expressions for \( l_A \) and \( l_B \). We obtain:

\[
l_A = \frac{-16A^2 + 16AB - 4B^2 + 8A - 4B - 36Ak + 27k^2 + 54k - 1}{72k} t
\]

(A.16)

\[
l_B = \frac{-16B^2 + 16AB - 4A^2 + 8B - 4A + 36Bk + 27k^2 + 18k - 1}{72k} t
\]

(A.17)
Now, using (A.17) and our assumption that $l_B$ is equal to the candidate equilibrium price, $l_B = l$, we can obtain $B$:

$$B = \frac{1}{4} + \frac{1}{2}A + \frac{9}{8}k - \frac{3}{32}\sqrt{178k^2 - 136k + 8 + 128Ak - (4 - 2k)\sqrt{97k^2 - 68k + 4}}$$  \hspace{1cm} (A.18)$$

Next, we substitute (A.16) and (A.18) into (A.7), so that the profit of firm $A$ becomes a function of only $A$. The result is the following:

$$\Pi_A = -\frac{5}{16k}A^3 + \frac{-1152\delta - 9728k + 11264}{215k}A^2t + \frac{640(k + 1)\delta + 1296k^2 - 3136k - 3520 - 16(2 - k)d}{215k}A^t$$

$$+ \frac{(-147k^2 + 172k - 3kd + 6d - 44)\delta - 276k^3 + 9512k^2 + 21200k + 96 + (-84k^2 + 128k + 80)d}{215k}t$$

where $\delta = \sqrt{178k^2 - 136k + 8 + 128Ak - (4 - 2k)d}$, and $d = \sqrt{97k^2 - 68k + 4}$.

We will not put any more algebraic details here. In fact, $\Pi_A$ would be a cubic polynomial of $A$, if the term $\delta$ did not have $A$ in it. With this $\delta$ added, $\Pi_A$ is equivalent to a 6th-order polynomial of $A$, with negative coefficient on the highest order term. Therefore, the global maximum must be one of the roots of the first derivative. The first derivative is a 5th-order polynomial of $A$. This has five roots. One of the roots is, of course $A = \frac{3}{8} - \frac{7}{16}k + \frac{1}{16}\sqrt{97k^2 - 68k + 4}$. Our numerical calculations have shown that for all $k$’s in the relevant range, two roots are always complex, and the remaining two real roots are “too small”, i.e. with them $A + \frac{5}{8} < \frac{1}{2}$. Therefore, we can safely assert now, that the global maximum is either the candidate solution or some sort of corner solution. There are two types of corner solution in this problem, the “short-term contracts only from firm $A$”, and the “long-term contracts only from firm $A$.” Lemma 6 and Lemma 7 establish the conditions, under which these deviations are worse for firm $A$ than sticking to the candidate equilibrium strategy.

Below, in Lemma 4, we establish the equilibrium outcome of the second-period game, given that all four contracts are sold in the first period. This outcome, of course, is conditional on the outcome of the first period. In particular, it depends on how many consumers have bought long-term contracts in the first period.

**Lemma 4** If the first-period cutoffs, $A$ and $B$ are such that $A \geq |\frac{1}{2} - \frac{k}{2}|$, and $1 - B \geq |\frac{1}{2} - \frac{k}{2}|$ then the second period prices and the cutoff are as follows:
\[ s_A = \left( \frac{1 - A - B}{3} + k \right) t \]  
(A.19)

\[ s_B = \left( \frac{A + B - 1}{3} + k \right) t \]  
(A.20)

\[ S = \frac{2A + 2B + 1}{6} \]  
(A.21)

**Proof.** As in the textbook Hotelling model, there will be a cutoff level \( S \) within the interval \( [A, B] \), such that consumers choose one firm or the other depending whether they are to the left or to the right of \( S \). This cutoff is the location of the marginal consumer, who is indifferent between the two firms. It is given by the well-known formula: \( S = \frac{1}{2} + \frac{s_A + s_B}{2t} \). Now, the profits from the second-period contracts are given by the following expressions:

\[
\Pi_A^2 = s_A \int_A^B F(S|_x_1) \, dx_1 = s_A \int_A^B \left( \frac{S - x_1}{k} + \frac{1}{2} \right) \, dx_1 = s_A \frac{(B - A)(2S - B - A + k)}{2k}
\]

\[
\Pi_B^2 = s_B \int_A^B (1 - F(S|_x_1)) \, dx_1 = s_B \int_A^B \left( 1 - \frac{S - x_1}{k} - \frac{1}{2} \right) \, dx_1 = s_B \frac{(B - A)(B + A - 2S + k)}{2k}
\]

After calculating the integrals, and substituting in the term \( S \), we get the profits as only functions of the prices \( s_A \) and \( s_B \). It turns out that these are globally concave functions of the own prices, so the first-order conditions are sufficient. Below we give these expressions:

**FOC:**

\[
\frac{\partial}{\partial s_A} \Pi_A^2 = \frac{(B - A - B^2 + A^2 + Bk - Ak) t - (B - A)(2s_A - s_B)}{2kt} = 0
\]

\[
\frac{\partial}{\partial s_B} \Pi_B^2 = \frac{-(B - A - B^2 + A^2 - Bk + Ak) t - (B - A)(2s_B - s_A)}{2kt} = 0
\]

**SOC:**

\[
\frac{\partial^2}{\partial^2 s_A} \Pi_A^2 = -\frac{B - A}{kt} < 0
\]

\[
\frac{\partial^2}{\partial^2 s_B} \Pi_B^2 = -\frac{B - A}{kt} < 0
\]

The prices stated above form the unique solution to the first-order conditions.  

**Lemma 5** If \( k \in [2/3, 1) \) then the following prices and cutoff levels are not an equilibrium: \( l_A = l_B = \frac{t + kt}{2}, f_A = f_B = t, s_A = s_B = kt, A = \frac{1}{2} - \frac{k}{2}, B = \frac{1}{2} + \frac{k}{2} \).

**Proof.** First of all, these prices and cutoff levels could constitute an equilibrium only if \( k < 1 \). Otherwise, \( A \) would be smaller than 0 and \( B \) would be greater than 1, which do not make sense in our
model. In this candidate equilibrium, the profit of firm A would be the following: $\Pi^q_A = \frac{4 + k}{2}$. We show that there exists a deviation, which yields larger profit. Consider that A offers only short-term contracts, with $f_A = t$. So, by definition $A = 0$. We prove that in this case, although firm B offers long-term contracts, no customer would buy them, so $B = 1$. To see this, substitute the candidate prices into (A.10), after simplifications we get: $g^B(A, B, l_B) = \frac{(16B^2 - 36k + 9k^2 + 18k + 1)t}{36k}$. This formula is a decreasing function of $B$ for all $k \in [2/3, 1]$. In addition, at $B = 1$ it is equal to $\frac{(1-k)^2}{4k}$, which is positive. So, it will be positive for all $B$. In words this means that for all customers, the expected cost of the long-term contract is higher than that of the short-term contract. Hence, $B = 1$. In this case, since nobody buys long-term contracts in the first period, nobody is committed to a particular firm in the second period. So, just as in Proposition 3, the game simplifies to a repetition of two one-shot games, with no intertemporal linkages. The solution is $f_A = f_B = s_A = s_B = t$. The resulting profit is: $\Pi^q_A = t$, which is larger than $\Pi^q_A$ for all $k < 1$.

**Lemma 6** If firm B offers $l_B = \frac{79k^2 + 164k - 4(2 - k)\sqrt{97k^2 - 68k + 4}}{256k}$ and $f_B = t$ then A would not deviate to the strategy of offering short-term contracts only, provided that $k \geq 0.81$ (approximately).

**Proof.** (See lemma6.mws) By charging the candidate equilibrium prices, firm A’s profit would be:

$$\Pi^q_A = \frac{123k^3 + 370k^2 + 564k - 8 - (21k^2 - 8k - 4)\sqrt{97k^2 - 68k + 4}}{1024k}$$

Suppose that firm A deviates, by offering only short-term contracts. It is easy to show that his optimal short-term price would still be $f_A = t$. It turns out, that in this case we can solve for the cutoffs in closed form. Obviously, $A = 0$, by assumption. Then from (A.10) we can express $B$. Since (A.10) is a quadratic equation in $B$, it has two roots, but it’s easy to see that one of them is greater than 1, if $k \geq \frac{3}{8}$. Therefore, we consider the smaller root, $B = \frac{8 + 36k - 3\sqrt{178k^2 - 136k + 8 - (4 - 2k)d}}{32}$. Substituting this into (A.7) yields:

$$\Pi^q_A = \frac{(-147k^2 + 172k - 44 + 3(2 - k)d)\delta' - 276k^3 + 9512k^2 + 21200k + 96 - (84k^2 - 128k - 80)d}{216k}$$

where $\delta' = \sqrt{178k^2 - 136k + 8 - (4 - 2k)d}$ and $d = \sqrt{97k^2 - 68k + 4}$. Both $\Pi^q_A$ and $\Pi^{dev}A$ are strictly increasing in $k$ for $k$ in the relevant range. It is a fairly straightforward exercise to show that there exists a $k^*$, such that $\Pi^q_A \geq \Pi^{dev}A$ if and only if $k \geq k^*$. This threshold level turns out to be one of the roots of the following 8th-order polynomial: $38046501Z^8 - 80585340Z^7 - 9011246Z^6 + 63690268Z^5 - 1313407Z^4 - 11634304Z^3 - 2170424Z^2 - 127232Z - 2352 = 0$. Numerically, this $k^*$ is approximately 0.81.

**Lemma 7** If firm B offers $l_B = \frac{79k^2 + 164k - 4(2 - k)\sqrt{97k^2 - 68k + 4}}{256k}$ and $f_B = t$ then A would not deviate to the strategy of offering long-term contracts only.
Proof. (See lemma7.mws) By charging the candidate equilibrium prices, firm A’s profit would be:

$$\Pi^e_A = \frac{123k^3 + 370k^2 + 564k - 8 - (21k^2 - 8k - 4)\sqrt{(97k^2 - 68k + 4)}}{1024k}t$$

Suppose that firm A makes a deviation, such that he only offers long-term contracts. Then we have to modify the equation that determines the cutoff $A$. In the symmetric equilibrium of Proposition 4, it was the location of the customer who is indifferent between $LT_A$ and $ST_A$. In the current case, it will be the location of the customer who is indifferent between $LT_A$ and $ST_B$. The expected payments and the difference in transportation costs between the two contracts are the following:

$$P_{LT_A} = 2l_A$$
$$P_{ST_B} = f_B + s_A F(S|A) + s_B (1 - F(S|A))$$
$$\Delta T(LT_A; ST_B) = (1 - 2A) t + t \int_S^{A + \frac{1}{2}} (2x_2 - 1) f(x_2|A) dx_2$$

With this, equation (A.9), the one which determines $A$ becomes:

$$g^A(A, B, l_A) \equiv 2l_A + t(1 - 2A) - f_B - s_A \left(\frac{S + A}{2k} + \frac{1}{2}\right) - s_B \left(\frac{S + A}{2k} + \frac{1}{2}\right) + \frac{1}{2} (A^2 - A - S^2 + S + Ak + \frac{1}{4}k^2 - \frac{1}{2}k) = 0$$

As a consequence, $l_A$ will be slightly different from (A.16).

$$l_A = \frac{-16A^2 + 16AB - 4B^2 + 8A - 4B - 108Ak + 27k^2 + 90k - 1}{72k}$$

In addition, the profit function needs a slight modification, as the term $f_A \left(\frac{1}{2} - A\right)$ is no longer there:

$$\Pi^{\text{dev}}_A = 2Al_A + s_A (\frac{B - A + 1}{2k} \cdot (2A - B - A + k))$$

Everything else carries through exactly the way it did in the sufficiency part of Proposition 4. For example, $B$ is still given by equation (A.18). Then, one can substitute $l_A$, and $B$ into the profit function, which will be only a function of $A$.

$$\Pi^{\text{dev}}_A = \frac{-5A^3t + \frac{-1152x - 75264k + 11264}{215k}A^2t + \frac{640(k + 1)x + 1296k^2 + 62400k - 3520 - 16(2 - k)d}{215k}At}{215k} + \frac{\delta - 276k^3 + 9512k^2 + 4816k + 96 + (-84k^2 + 128k + 80)}{215k}d$$

where $\delta = \sqrt{178k^2 - 136k + 8 + 128Ak - (4 - 2k)d}$, and $d = \sqrt{97k^2 - 68k + 4}$.

This profit function is very similar to the one in the sufficiency part of Proposition 4. Also, similarly to
that one, this formula as well is equivalent to a 6th-order polynomial of \( A \). Moreover, the coefficient of the highest order term is negative. Therefore, the global maximum must be one of the local maxima (there are at most three local maxima). Our numerical calculations have shown that for all \( k \in [2/3, 1/4 + \sqrt{17}/4] \), two roots of the first-order condition are complex. Then we tried the other three roots, which were real. For all of them, \( \Pi_{A}^{\text{lev}} \) was strictly less than \( \Pi_{A}^0 \). ■

**Proposition 5** If \( 1.25 \leq k \leq 2 \), then there exists a symmetric equilibrium, in which firms offer only long-term contracts. The equilibrium prices are the following:

\[
l_{A} = l_B = \frac{t}{k}
\]

**Proof.** (See prop3.mws) First, we prove the necessity part, i.e. we show that if only long-term contracts were allowed then the equilibrium prices would be the ones stated above. In this setting, customers have only two options, so there will be only one cutoff, which we will denote by \( C \). The expected payments and the difference in transportation costs between the two contracts are the following:

\[
\begin{align*}
P_{LT_A} &= 2l_A \\
P_{LT_B} &= 2l_B \\
\Delta T (LT_A; LT_B) &= (2C - 1)t + \frac{t}{k} \left( \int_{C - \frac{t}{k}}^{0} (-2x_2 - 1) \, dx_2 + \int_{0}^{1} (2x_2 - 1) \, dx_2 + \int_{1}^{C + \frac{t}{k}} (3 - 2x_2) \, dx_2 \right)
\end{align*}
\]

Now, if we calculate the integrals, we get: \( P_{LT_A} - P_{LT_B} + \Delta T (LT_A; LT_B) = \frac{2(l_A - l_B)k + (2C - 1)t}{k} \). By construction, this is equal to zero. Thus, we can express the cutoff: \( C = \frac{1}{2} - \frac{(l_A - l_B)k}{2t} \). Substitute this into the profit functions:

\[
\begin{align*}
\Pi_A &= 2l_AC = 2l_A \left( \frac{1}{2} - \frac{(l_A - l_B)k}{2t} \right) \\
\Pi_B &= 2l_B(1 - C) = 2l_B \left( \frac{1}{2} + \frac{(l_A - l_B)k}{2t} \right)
\end{align*}
\]

These functions are globally concave in the own prices, because the second derivatives are negative. \( \frac{\partial^2}{\partial l_A^2} \Pi_A = \frac{\partial^2}{\partial l_B^2} \Pi_B = -\frac{2k}{t} < 0 \). Therefore the unique equilibrium can be found by setting the first-order conditions to zero. The solution is the one stated above.

To prove sufficiency, we must show that firms do not have a profitable deviation, by offering short-term contracts as well. Suppose that firm \( B \) offers only long-term contracts and sets its price to be \( l_B = \frac{t}{k} \). We show that the best that firm \( A \) could do is to do the same. Suppose \( A \) deviates, by offering both
types of contracts at appropriate prices that both contracts are bought by some customers.\textsuperscript{17} As usual, the
cutoff $A$ is the location of the marginal customer who is indifferent between $LT_A$ and $ST_A$. However, now
the customer at $B$ is the one who is indifferent between $ST_A$ and $LT_B$, since $B$ does not offer short-term
contracts. As usual, we write the expected costs of these contracts as follows:

$$
\begin{align*}
P_{ST_A} &= f_A + s_A \left( \frac{S - B}{k} + \frac{1}{2} \right) + s_B \left( \frac{-S + B}{k} + \frac{1}{2} \right) \\
P_{LT_B} &= 2l_B \\
\Delta T(ST_A; LT_B) &= (2B - 1) t + \frac{t}{k} \left( \int_{B - \frac{1}{2}}^{0} (-2x_2 - 1) dx_2 + \int_{0}^{S} (2x_2 - 1) dx_2 \right)
\end{align*}
$$

Notice that, in this case it is not true anymore that $A$’s optimal first-period price should be $f_A = t$, it
would only be true if $B$ offered a short-term contract and set its price equal to $f_B = t$. So, the profit of
firm $A$ is the following:

$$
\Pi_A = 2Al_A + f_A (B - A) + s_A \frac{(B - A) (2S - B - A + k)}{2k}
$$

Now, the equations that determine the first-period cutoffs, after simplification are:

$$
\begin{align*}
g^A (A, B, l_A, f_A) &= \left( -56A^2 - 16AB + 4B^2 + 136A + 4B - 364k - 45k^2 + 54k - 71 \right) t + 72l_A k - 36f_A k = 0 \\
g^B (A, B, f_A) &= \left( 4A^2 - 16AB - 56B^2 + 4A - 8B - 36Bk - 45k^2 + 54k + 73 \right) t - 36f_A k = 0
\end{align*}
$$

Now, if we solve these equations for $l_A$ and $f_A$, we get the following:

$$
\begin{align*}
l_A &= \frac{5A^2 - 5B^2 - 11A - B + 3Ak - 3Bk + 12}{6k} t \\
f_A &= \frac{4A^2 - 16AB - 56B^2 + 4A - 8B - 36Bk - 45k^2 + 54k + 73}{36k} t
\end{align*}
$$

Then, if we substitute these into the profit function, after some rearrangement we arrive at the following
expression:

$$
\Pi_A = \frac{(A - B) \left( 12 (A + B)^2 + 6 (A^2 + B^2) + (16k - 20) (A + B) + 9k^2 - 22k - 1 \right) + 24 (A - A^2 + B - B^2)}{12k} t
$$

\textsuperscript{17}In a sense, this case is just the opposite to the one in Lemma 7.
If there was an interior solution (meaning $0 < A < B < 1$) to this maximization problem, then $\frac{\partial}{\partial A} \Pi_A = 0$, and $\frac{\partial}{\partial B} \Pi_A = 0$. As a consequence, also $\frac{\partial}{\partial A} \Pi_A + \frac{\partial}{\partial B} \Pi_A = 0$. Evaluate the sum of the two first-order conditions. After simplifying, we get:

$$\frac{\partial}{\partial A} \Pi_A + \frac{\partial}{\partial B} \Pi_A = \frac{(A - B)(15(A + B) + 8k - 10) - 12(A + B)}{3k} t$$

This expression is negative if $k \geq \frac{5}{4}$ and $A < B$. Therefore, in this case, there can’t be an interior solution, which implies that at the maximum, $A = B$. Then, the profit function simplifies to $\Pi_A = \frac{4A(1-A)}{k} t$. The maximizer to this expression is $A = \frac{1}{2}$. Hence, the best response of firm $A$ is to charge the same price as $B$. ■